# Milestone 04

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## Shear-wave decay

### Introduction

In this milestone we will take a look at a method commonly used in computational physics to measure the kinematic viscosity of a fluid. The shear wave method consists in setting a sinusoidal velocity profile in our box and then measuring how fast it decays.

As a brief reminder, the discrete Boltzmann transport equation reads

$$f_i(\mathbf{r}+\Delta t\mathbf{c}_i, t+\Delta t) = f_i(\mathbf{r},t) + \omega\left(f_i^{eq}(\mathbf{r},t) - f_i(\mathbf{r},t)\right)$$

In the previous milestone we identified how $f_i^{eq}(\mathbf{r})$ looks like. Instead of measuring $\rho$ and $\mathbf{u}$ to enter the equilibrium distribution function

$$f_i^{eq}(\rho(\mathbf{r}),\mathbf{u}(\mathbf{r})) = w_i\rho\left[1 + 3\mathbf{c}_i\cdot\mathbf{u}(\mathbf{r}) + \frac{9}{2}\left(\mathbf{c}_i\cdot\mathbf{u}(\mathbf{r})\right)^2 - \frac{3}{2}\mathbf{u}(\mathbf{r})^2\right]$$

with

$$w_i = \begin{cases} \frac{4}{9} & \text{for } i=0 \\ \frac{1}{9} & \text{for } i\in\{1,2,3,4\} \\ \frac{1}{36} & \text{for } i\in\{5,6,7,8\} \end{cases}$$

we now *prescribe* the initial values of $\rho$ and $\mathbf{u}$ at time $t=0$ and initialize the populations with the corresponding equilibrium distribution, $f_i(\mathbf{r},0) = f_i^{eq}(\rho(\mathbf{r}),\mathbf{u}(\mathbf{r}))$.

## Measuring the viscosity

The shear-wave decay method starts from an initial *shear wave*, a sinusoidal velocity perturbation that decays over time due to viscosity. For an initial perturbation $u_x(y, 0) = \varepsilon \sin(ky)$ with wave vector $k = 2\pi/n_y$, the linearized Navier-Stokes equations predict that the wave keeps its sinusoidal shape while its amplitude decays exponentially,

$$u_x(y,t) = \varepsilon\, e^{-\nu k^2 t} \sin(ky), \tag{1}$$

where $\nu$ is the kinematic viscosity. This gives a simple measurement procedure: record the amplitude $\hat{u}_x(t)$ of the sine wave (e.g. the velocity at the position of the initial maximum, or the amplitude of the corresponding Fourier mode) at every time step, plot $\ln \hat{u}_x$ versus $t$, and fit a straight line. The slope of this line is $-\nu k^2$, from which you obtain $\nu$. The full measurement procedure, including the derivation of Equation 1, is described in the lecture notes on shear-wave decay.

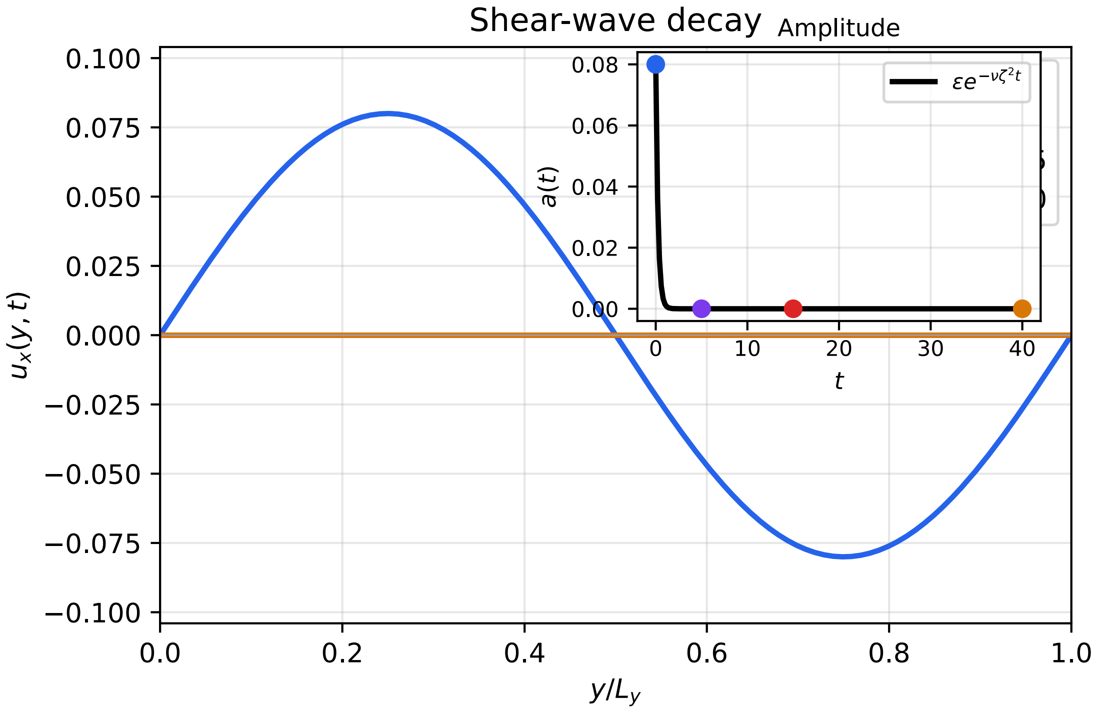


Figure 1: Decay of a shear wave: the sinusoidal velocity profile $u_x(y)$ keeps its shape while its amplitude decays exponentially in time with rate $\nu k^2$.

## Tasks

- Choose an initial distribution of $\rho(\mathbf{r}) = 1$ and $u_x(\mathbf{r}) = \varepsilon \sin\left(\frac{2\pi y}{n_y}\right)$ at $t = 0$, i.e. a sinusoidal variation of the velocities $u_x$ with the position $y$.
- Observe what happens dynamically as well as in the long time limit $t \to \infty$.
- Assume that the initial conditions fulfill the Stokes flow condition, i.e.

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \Delta \mathbf{u}$$

and calculate the kinematic viscosity.

## Notes

- Choose $\omega$ with care, i.e. $0 < \omega < 2$.
- Choose $0 < \rho < 1$ and $|\mathbf{u}| < 0.1$.
- For the fulfillment of the Stokes condition choose $\varepsilon$ with care.
- Your final presentation should contain plots of the evolution of the velocity profile with time. All plots must follow our figure rules.
- The presentation should also report on the measured viscosity as a function of the parameter $\omega$. The analytical relationship between the kinematic viscosity and the relaxation parameter (derived from the Chapman-Enskog expansion) is

  $$\nu = \frac{1}{3}\left(\frac{1}{\omega} - \frac{1}{2}\right).$$

  Produce a plot that shows your numerical measurements alongside this analytical prediction.