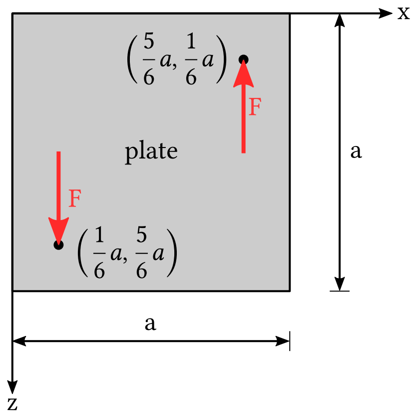
# Exercise 1: Forces and moments
## 18.10.2021 - 22.10.2021

**Question 1** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
The picture below shows a rigid plate. Two forces of equal magnitude $F$ are applied in opposite direction at the indicated points.



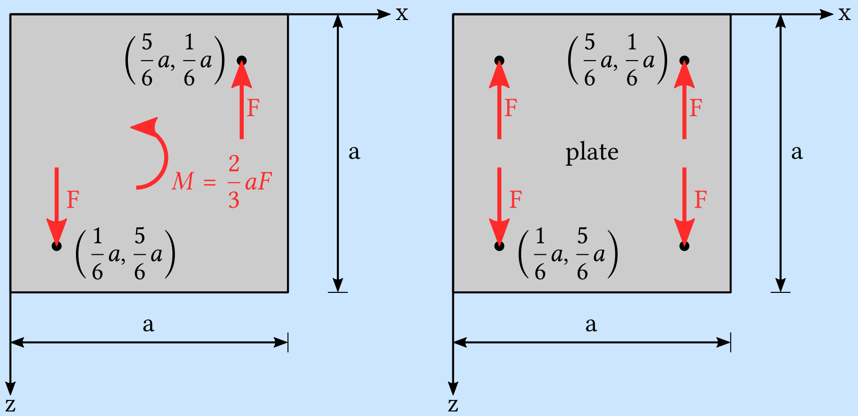

(a) Is the system in static equilibrium?

(b) If you find that the system is not in static equilibrium, then determine the forces that would need to be applied in order to achieve equilibrium!

**Solution:**

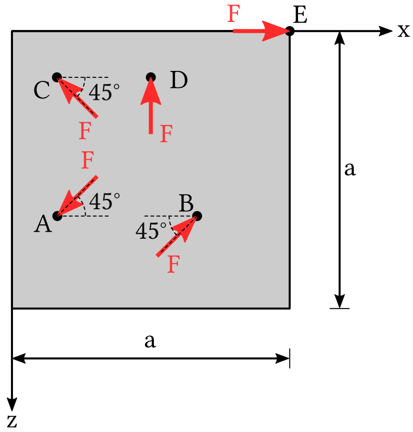
(a) No, the system is not in static equilibrium. The resultant force is zero, but the two forces generate a moment $M = 2/3aF$ in the positive (counter-clockwise) sense about the $y$-axis (pointing out of the plane of the paper). Since the plate is not fixed, it would start to rotate counter-clockwise.

(b) In order to achieve equilibrium we need to add a force couple that creates a moment of equal magnitude but opposite sense. For example, we could add a force $(0, 0, -F)$ at $x = a/6, z = a/6$ and another force $(0, 0, F)$ at $x = 5a/6, z = 5a/6$.



**Question 2** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

The rigid plate shown below is subjected to five point forces at the indicated points. All forces have the same magnitude $F$, but their orientation differs. Find the resultant force and moment, and indicate them in the picture!




$$A = \left(\frac{1}{6}a, \frac{2}{3}a\right)$$

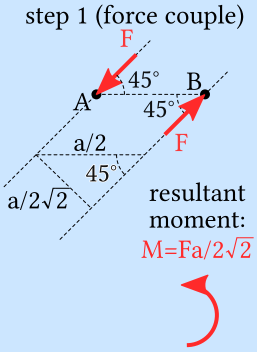
$$B = \left(\frac{2}{3}a, \frac{2}{3}a\right)$$

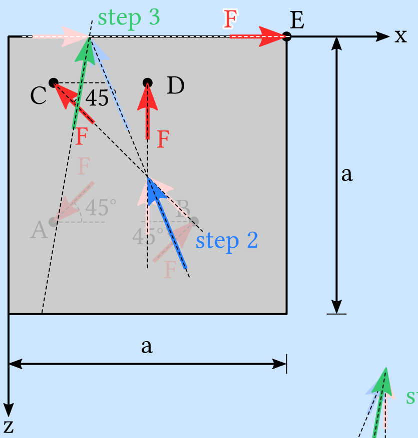
$$C = \left(\frac{1}{6}a, \frac{1}{6}a\right)$$

$$D = \left(\frac{1}{2}a, \frac{1}{6}a\right)$$

$$E = (a, 0)$$

**Solution:**




$$A = \left(\frac{1}{6}a, \frac{2}{3}a\right)$$

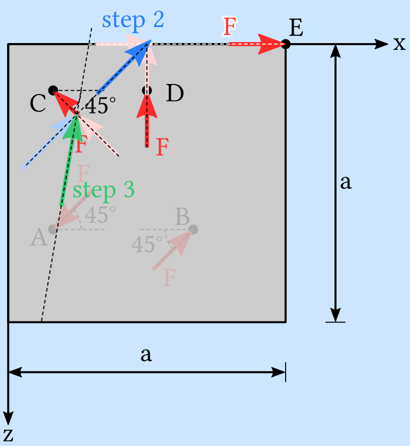
$$B = \left(\frac{2}{3}a, \frac{2}{3}a\right)$$

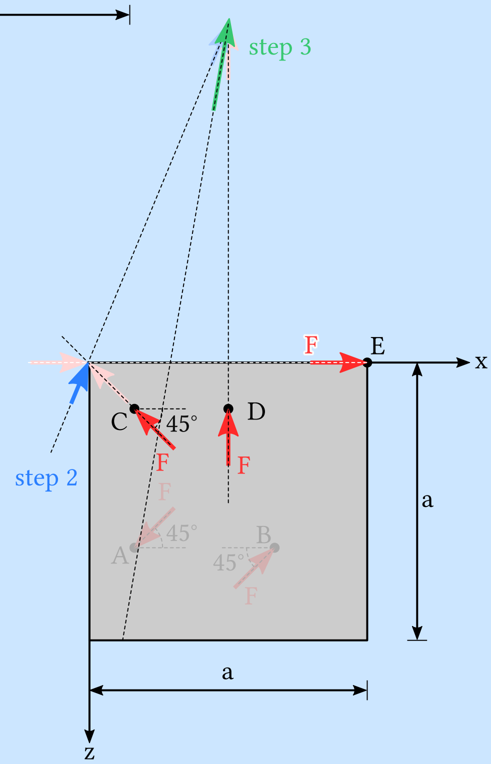
$$C = \left(\frac{1}{6}a, \frac{1}{6}a\right)$$

$$D = \left(\frac{1}{2}a, \frac{1}{6}a\right)$$

$$E = (a, 0)$$

alternatives for steps 2 and 3:



**step 1** The forces at $A$ and $B$ are equal and opposite. Therefore, their contribution to the resultant force is zero. However, they create a positive (counter-clockwise) moment about the $y$-axis (pointing out of the plane of the paper). The distance between the lines of actions is $a/(2\sqrt{2})$, therefore the magnitude of the moment is $M = Fa/(2\sqrt{2})$.

**step 2** The force vectors at $C$ and $D$ are

$$\vec{F}_C = \begin{pmatrix} -F/\sqrt{2} \\ 0 \\ -F/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \vec{F}_D = \begin{pmatrix} 0 \\ 0 \\ -F \end{pmatrix}.$$

Adding them yields the intermediate result

$$\vec{F}_2 = \begin{pmatrix} -F/\sqrt{2} \\ 0 \\ -F(1+1/\sqrt{2}) \end{pmatrix}.$$

To find $\vec{F}_2$ graphically, we first move $\vec{F}_C$ and $\vec{F}_D$ along their lines of action to the point where these lines intersect. Then we add $\vec{F}_C$ and $\vec{F}_D$ head to tail. In the figure above, the lines of action are shown as dashed lines. $\vec{F}_2$ is the blue vector.

**step 3** In order to find the resultant force, we repeat the same procedure with $\vec{F}_2$ and the force at $E$

$$\vec{F}_E = \begin{pmatrix} F \\ 0 \\ 0 \end{pmatrix}.$$

The resultant force is

$$\vec{F}_3 = \vec{F}_2 + \vec{F}_E = \begin{pmatrix} F(1-1/\sqrt{2}) \\ 0 \\ -F(1+1/\sqrt{2}) \end{pmatrix}.$$

$\vec{F}_3$ is the green vector in the figure above.

For the graphical solution, it does not make a difference whether we start with $\vec{F}_C$ and $\vec{F}_D$, or with $\vec{F}_D$ and $\vec{F}_E$, or with $\vec{F}_C$ and $\vec{F}_E$. We will obtain a force vector with the same length, orientation, and line of action, see the two lower figures.

What is the formula for the line of action of the resultant force $\vec{F}_3$? To answer this question, we first need to find the line of action of $\vec{F}_2$. Note that the lines of action of $\vec{F}_C$ and $\vec{F}_D$ intersect at $x = a/2$, $z = a/2$. The line of action of $\vec{F}_2$ can therefore be parameterized as

$$\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = a \begin{pmatrix} 1/2 \\ a/2 \end{pmatrix} + t \cdot a \begin{pmatrix} -1/\sqrt{2} \\ -1-1/\sqrt{2} \end{pmatrix}, \quad t \in \mathbb{R}.$$

The line of action of $\vec{F}_E$ goes through the origin and can therefore be parameterized as

$$\begin{pmatrix} x(s) \\ z(s) \end{pmatrix} = s \cdot a \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}.$$

To find the point of intersection of the two lines, we need to find $s'$ and $t'$ such that

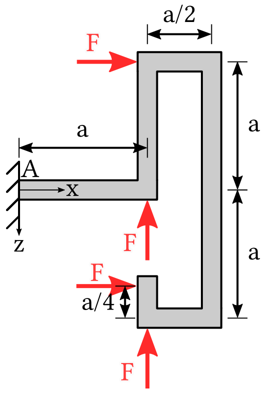
$$a \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + t' \cdot a \begin{pmatrix} -1/\sqrt{2} \\ -1-1/\sqrt{2} \end{pmatrix} = s' \cdot a \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The second equation yields $t' = 1/(2+\sqrt{2})$. By inserting this result into the first equation, we get $s' = t' = 1/(2+\sqrt{2})$. Thus, the point of intersection is at $x = a(1-1/(1+\sqrt{2}))/2$, $z = 0$. Together with the result for $\vec{F}_3$, we can now write the following parameterization for the line of action of $\vec{F}_3$,

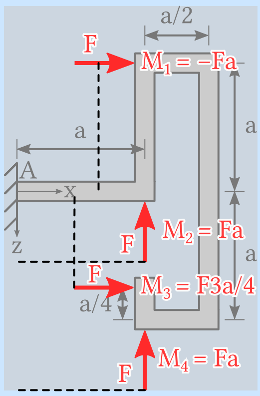
$$\begin{pmatrix} x(r) \\ z(r) \end{pmatrix} = a \begin{pmatrix} 1/(2+\sqrt{2}) \\ 0 \end{pmatrix} + r \cdot a \begin{pmatrix} 1-1/\sqrt{2} \\ -1-1/\sqrt{2}) \end{pmatrix}, \quad r \in \mathbb{R}.$$

**Question 3** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

A bar with multiple corners is fixed on a wall and loaded by four forces of magnitude $F$. Sort the forces according to the resulting bending moment about the $y$-axis in point $A$, from highest to lowest!




**Solution:**

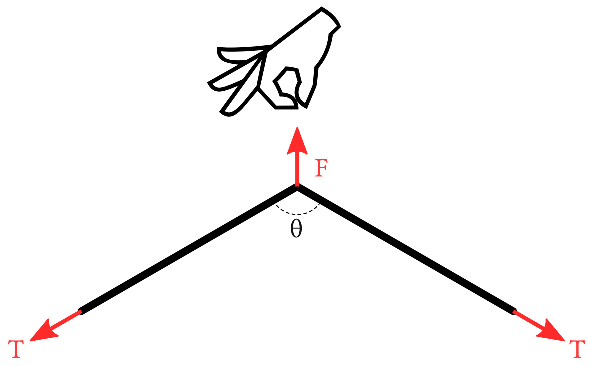



In the figure above, the dashed lines indicate the lever arms of the forces. Based on these distances, we obtain the moments.

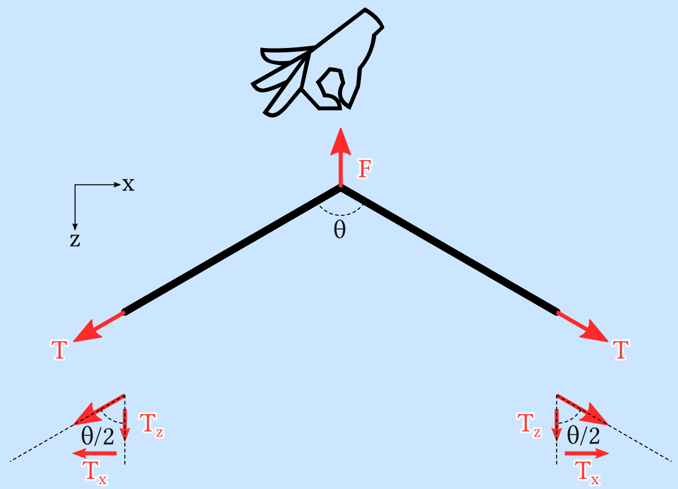
$$M_1 = -Fa, \quad M_2 = Fa, \quad M_3 = \frac{3}{4}Fa, \quad M_4 = Fa.$$

The positive sense of the moment is counter-clockwise with respect to the $y$-axis, which is the axis coming out of the plane of the paper (recall the right hand rule!). Thus, if we want to sort the moments from highest to lowest, accounting for the sign, we should write $M_2 = M_4 > M_3 > M_1$.

**Question 4** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
A string is plucked with a force $F$ at the middle of the string. Calculate the line tension $T$ that needs to act in the string so that the system is in equilibrium!




**Solution:**

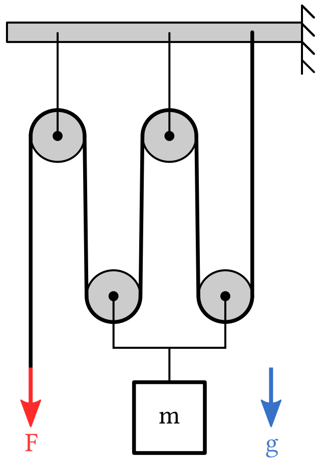



We introduce a coordinate system and decompose $T$ into its $x$- and $z$-components $T_x$ and $T_z$, respectively. From geometry $T_x = \sin(\theta/2)T$ and $T_z = \cos(\theta/2)T$. For the system to be in equilibrium, the force $F$ must be balanced by the two components $T_z$, i.e.

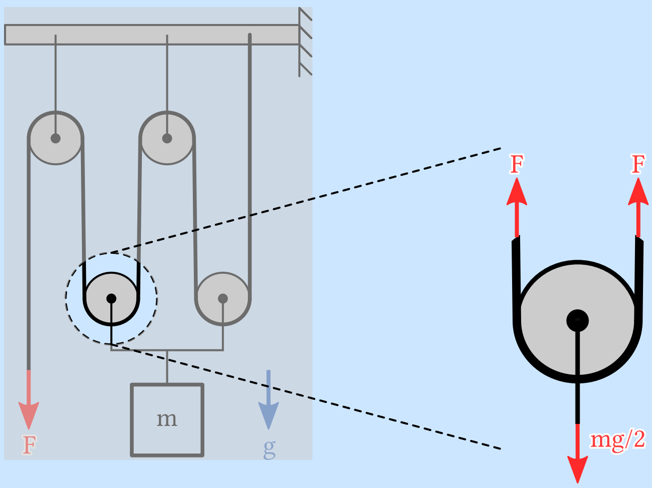
$$-F + 2T_z = 0 \leftrightarrow -F + 2\cos(\theta/2)T = 0 \implies T = \frac{F}{2\cos(\theta/2)}.$$

**Question 5** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

A rigid body of mass $m$ is suspended from a system of pulleys in the gravitational field of earth (constant acceleration $g$). Calculate the force $F$ that you need to apply at the left end of the rope to hold the mass in position!




**Solution:**




The gravitational force on the mass is $mg$. The mass is connected to two pulleys, so each pulley receives $mg/2$. We now consider the second pulley from the left. We make hypothetical cuts through the rope on either side of the pulley. When making each cut, we simultaneously introduce a new external force, which acts on the rope at the point of the cut and equals the internal force at the point of the cut in the whole system. In this way, we preserve loading conditions on the cutout. The internal force in the rope is $F$. It is constant throughout the rope. Thus the newly introduced force is $F$. If the system is in equilibrium, then the two newly introduced forces must be balanced by the gravitational force, i.e.

$$2F - mg/2 = 0 \implies F = mg/4.$$