Exercise 3: Line loads and Dirac deltas

08.11.2024 - 11.11.2024

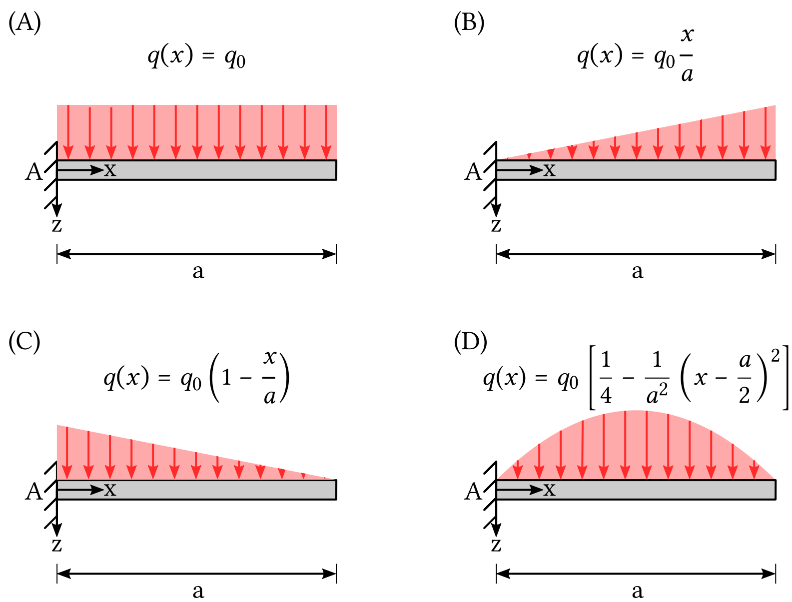
Question 1 . . . .

Figures (A)–(D) show a beam of length a which is mounted on a wall at point A. A line load is imposed on the beam.

(a) Calculate the bending moment about the y-axis and the force in z-direction at point A!

(b) Each line load can be replaced by an equivalent point load, which generates the same moment and force at the support. Find the magnitude and line of action of this force!

Note: Introducing an equivalent force as in part (b) is useful for calculating reaction forces at supports. However, such a replacement must not be made when calculating *internal* forces!



Solution:

(a) Forces and moments

The problem is statically determinate, therefore we can find the reactions at the support either (1) by requiring equilibrium between them and the resultants of the line load, or (2) by determining the internal forces and moments and evaluating them at $x = 0$. Note that both solutions require us to evaluate integrals, but in a different context. In method 1, we need integration to determine the resultant of the line load. In 2, we solve the differential equations $dQ(x)/dx = -q(x)$ and $d^2M(x)/dx^2 = -q(x)$, which we derived in class.

Method 1 In all four cases the beam is parallel to x. The applied line load is directed in positive z-direction and may vary along x. For the purpose of calculating the reaction forces, the line load is equivalent to an external point force in positive z-direction with the magnitude

$$\int_0^a q(x)dx.$$

The resultant moment is

$$-\int_0^a q(x)xdx.$$

The minus sign on the second integral reflects the fact that the resulting moment acts in clockwise (mathematically negative) direction about the y-axis. We need to compute the integrals above to find the reactions at A.

Let A_z be the force in z-direction at A, and M_A the moment about the y-axis. Their sense is indicated in the figure below. Computing the integrals and requiring equilibrium leads to the following solutions:

(A)

$$
\begin{aligned}
&\int_0^a q(x)dx = \int_0^a q_0 dx = q_0 a \\
&-\int_0^a q(x)x dx = -\int_0^a q_0 x dx = -\frac{1}{2}q_0 a^2 \\
\downarrow \qquad & -A_z + q_0 a = 0 \implies A_z = q_0 a \\
A \qquad & -M_A - \frac{1}{2}q_0 a^2 = 0 \implies M_A = -\frac{1}{2}q_0 a^2
\end{aligned}
$$

(B)

$$
\begin{aligned}
&\int_0^a q(x)dx = \int_0^a q_0 \frac{x}{a} dx = \frac{1}{2}q_0 a \\
&-\int_0^a q(x)x dx = -\int_0^a q_0 \frac{x^2}{a} dx = -\frac{1}{3}q_0 a^2 \\
\downarrow \qquad & -A_z + \frac{1}{2}q_0 a = 0 \implies A_z = \frac{1}{2}q_0 a \\
A \qquad & -M_A - \frac{1}{3}q_0 a^2 = 0 \implies M_A = -\frac{1}{3}q_0 a^2
\end{aligned}
$$

(C)

$$
\begin{aligned}
&\int_0^a q(x)dx = \int_0^a q_0 \left(1 - \frac{x}{a}\right) dx = \frac{1}{2}q_0 a \\
&-\int_0^a q(x)x dx = -\int_0^a q_0 \left(x - \frac{x^2}{a}\right) dx = -\frac{1}{6}q_0 a^2 \\
\downarrow \qquad & -A_z + \frac{1}{2}q_0 a = 0 \implies A_z = \frac{1}{2}q_0 a \\
A \qquad & -M_A - \frac{1}{6}q_0 a^2 = 0 \implies M_A = -\frac{1}{6}q_0 a^2
\end{aligned}
$$

(D)

$$
\begin{aligned}
&\int_0^a q(x)dx = \int_0^a q_0 \left[\frac{1}{4} - \frac{1}{a^2}\left(x - \frac{a}{2}\right)\right] dx = \frac{1}{6}q_0 a \\
&-\int_0^a q(x)x dx = -\int_0^a q_0 \left[\frac{x}{4} - \frac{x}{a^2}\left(x - \frac{a}{2}\right)\right] dx = -\frac{1}{12}q_0 a^2 \\
\downarrow \qquad & -A_z + \frac{1}{6}q_0 a = 0 \implies A_z = \frac{1}{6}q_0 a \\
A \qquad & -M_A - \frac{1}{12}q_0 a^2 = 0 \implies M_A = -\frac{1}{12}q_0 a^2
\end{aligned}
$$

Method 2 Recall the differential equations for the internal transversal force $Q(x)$ and the moment $M(x)$,

$$
\begin{aligned}
\frac{dQ}{dx} &= -q(x), \\
\frac{d^2 M(x)}{dx} &= -q(x).
\end{aligned}
$$

In problems (A)–(D) there are no discontinuities which would require us to partition the beam into sectors. Integrating the equation above will lead to two unknown constants C_1 and C_2. Problems (A)–(D) have the same set of boundary conditions, $Q(a) = 0$ and $M(a) = 0$. Below is the solution for the integrals and the constants with these boundary conditions. Evaluating $Q(x)$ and $M(x)$ at $x = 0$ gives the reaction force and moment at $x = 0$, respectively.

(A) Integration yields:

$$Q(x) = -q_0 x + C_1, \quad M(x) = -\frac{1}{2} q_0 x^2 + C_1 x + C_2.$$

Application of boundary conditions $Q(a) = 0$ and $M(a) = 0$ gives

$$C_1 = q_0 a, \quad C_2 = -\frac{1}{2} q_0 a^2.$$

Substituting C_1 and C_2, and evaluating at $x = 0$ gives

$$A_z = Q(0) = q_0 a, \quad M_A = M(0) = -\frac{1}{2} q_0 a^2.$$

(B) Integration yields:

$$Q(x) = -q_0 \frac{x^2}{2a} + C_1, \quad M(x) = -q_0 \frac{x^3}{6a} + C_1 x + C_2.$$

Application of boundary conditions $Q(a) = 0$ and $M(a) = 0$ gives

$$C_1 = \frac{1}{2} q_0 a, \quad C_2 = -\frac{1}{3} q_0 a^2.$$

Substituting C_1 and C_2, and evaluating at $x = 0$ gives

$$A_z = Q(0) = \frac{1}{2} q_0 a, \quad M_A = M(0) = -\frac{1}{3} q_0 a^2.$$

(C) Integration yields:

$$Q(x) = -q_0 \left(x - \frac{x^2}{2a} \right) + C_1, \quad M(x) = -q_0 \left(\frac{x^2}{2} - \frac{x^3}{6a} \right) + C_1 x + C_2.$$

Application of boundary conditions $Q(a) = 0$ and $M(a) = 0$ gives

$$C_1 = \frac{1}{2} q_0 a, \quad C_2 = -\frac{1}{6} q_0 a^2.$$

Substituting C_1 and C_2, and evaluating at $x = 0$ gives

$$A_z = Q(0) = \frac{1}{2} q_0 a, \quad M_A = M(0) = -\frac{1}{6} q_0 a^2.$$

(D) Integration yields:

$$Q(x) = -q_0 \left(\frac{x^2}{2a} - \frac{x^3}{3a^2} \right) + C_1, \quad M(x) = -q_0 \left(\frac{x^3}{6a} - \frac{x^4}{12a^2} \right) + C_1 x + C_2.$$

Application of boundary conditions $Q(a) = 0$ and $M(a) = 0$ gives

$$C_1 = \frac{1}{6} q_0 a, \quad C_2 = -\frac{1}{12} q_0 a^2.$$

Substituting C_1 and C_2, and evaluating at $x = 0$ gives

$$A_z = Q(0) = \frac{1}{6} q_0 a, \quad M_A = M(0) = -\frac{1}{12} q_0 a^2.$$

(b) Equivalent point force

Let R_q be the equivalent point force. It points in positive z-direction and its magnitude is $\int_0^a q(x)dx$, i.e. the integral which we evaluated previously. The distance x_q from A can be computed by requiring that the resultant moment $R_q x_q$ is balanced by M_A. This yields four cases,

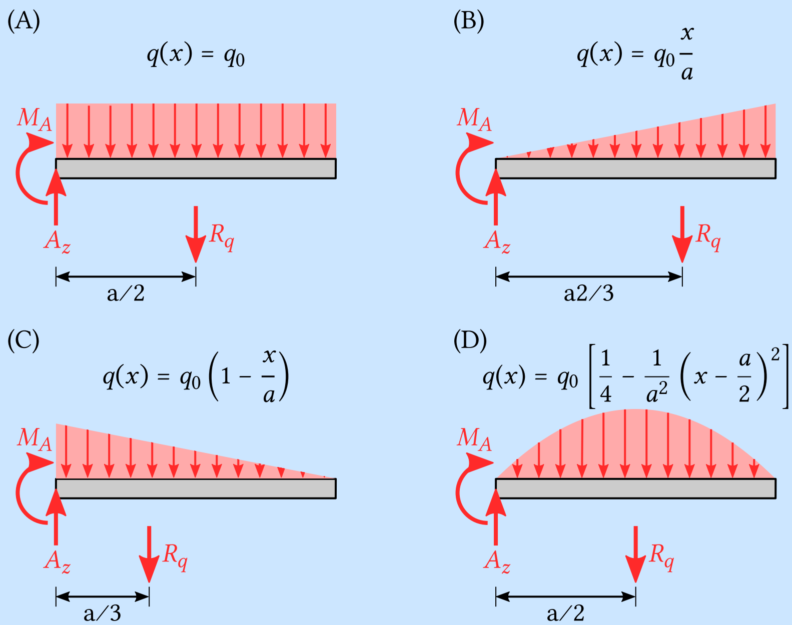
(A) $R_q = q_0 a \quad x_q = \frac{1}{2}a$

(B) $R_q = \frac{1}{2}q_0 a \quad x_q = \frac{2}{3}a$

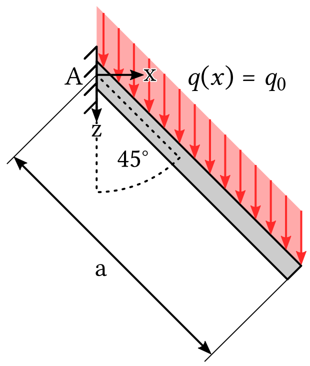
(C) $R_q = \frac{1}{2}q_0 a \quad x_q = \frac{1}{3}a$

(D) $R_q = \frac{1}{6}q_0 a \quad x_q = \frac{1}{2}a$

Note that x_q is the x-coordinate of the centroid of the line load.



Question 2 . . . .
A beam of length a is mounted on the wall at an angle of $45°$. A constant line load q_0 is applied. Calculate the reaction forces and the bending moment about the y-axis at point A!



Solution: To find the resultant of the line load, we need to integrate along the length of the beam. For this purpose, we introduce a coordinate system whose x'-axis is aligned with the beam axis and split the line load into parallel and perpendicular components $q_N(x')$ and $q_Q(x')$, respectively. Considering the angle, we obtain

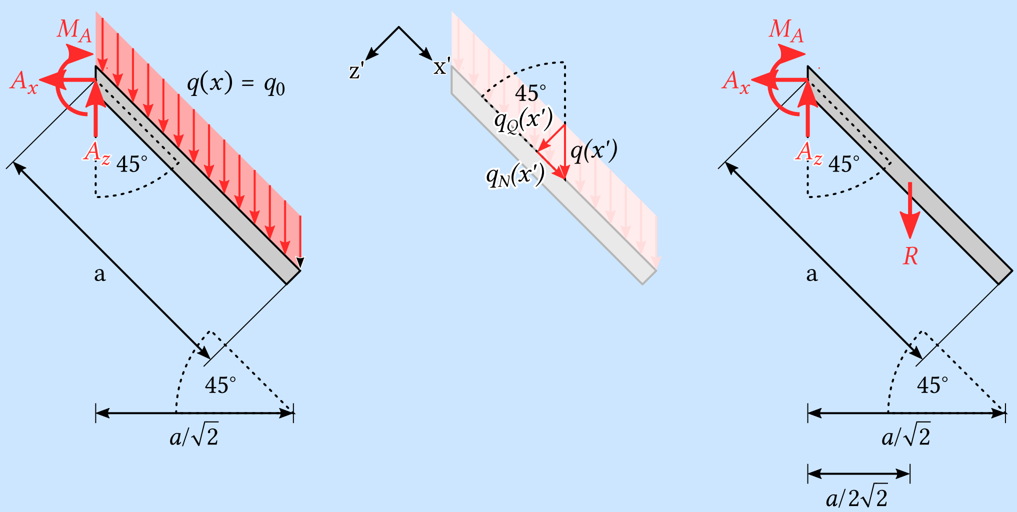
$$q_N(x') = q(x')\cos(45°) = q(x')/\sqrt{2} = q_0/\sqrt{2},$$
$$q_Q(x') = q(x')\sin(45°) = q_0/\sqrt{2}.$$

The resultant forces are

$$R_N = \int_0^a q_N(x')dx' = q_0 a/\sqrt{2}, \quad R_Q = \int_0^a q_Q(x')dx' = q_0 a/\sqrt{2}.$$

The magnitude of the resultant of $q(x')$ is therefore $R_q = \sqrt{R_N^2 + R_Q^2} = q_0 a$. It points along the z-direction of our original coordinate system. The center of mass of the line load lies halfway along the beam in x'-direction, i.e. at $x = a/(2\sqrt{2})$. Therefore, we can replace the line load by a resultant R acting at this point. Requiring equilibrium yields:

$$\begin{aligned}
&\rightarrow \quad && A_x = 0 \\
&\downarrow \quad && -A_z + R = 0 && \Longrightarrow A_z = q_0 a \\
&\text{A} \quad && -M_A - R\frac{a}{2\sqrt{2}} = 0 && \Longrightarrow M_A = -\frac{1}{2\sqrt{2}} q_0 a^2
\end{aligned}$$



Question 3 . . . .

Recall that we discussed the Dirac Delta function $\delta(x)$. Evaluate the following definite integrals!

(a) $\int_{-10}^{10} \left(x^2 - 2x + 1\right) \delta(x-2)dx$

(b) $\int_{-\infty}^{+\infty} \left(x^2 - 2x + 1\right) \delta(x+10)dx$

(c) $\int_{-\infty}^{+\infty} \left(f(x) - f(x_0)\right) \delta(x-x_0)dx$

Solution:

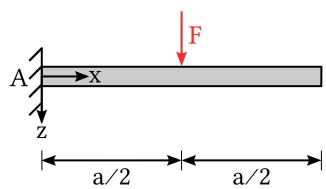
(a) $\int_{-10}^{10} \left(x^2 - 2x + 1\right) \delta(x-2)dx = 2^2 - 2 \cdot 2 + 1 = 1$

(b) $\int_{-\infty}^{+\infty} \left(x^2 - 2x + 1\right) \delta(x+10)dx = (-10)^2 + 20 + 1 = 121$

(c) $\int_{-\infty}^{+\infty} \left(f(x) - f(x_0)\right) \delta(x-x_0)dx = f(x_0) - f_(x_0) = 0$

Question 4 . . . .

A beam of length a is mounted on the wall. A force of magnitude F is applied in positive z-direction in the middle of the beam. Calculate the internal forces and moments *using integration*!



Solution: The point force can be described by the function $F\delta(x - \frac{a}{2})$, where δ is Dirac's Delta function. Let $Q(x)$ be the transversal force, then

$$\frac{dQ(x)}{dx} = -F\delta\left(x - \frac{a}{2}\right).$$

Integration yields

$$\int \frac{dQ(x)}{dx}dx = Q(x) = -FH\left(x - \frac{a}{2}\right) + C_1,$$
$$\int Q(x)dx = M(x) = -F\left(x - \frac{a}{2}\right)H\left(x - \frac{a}{2}\right) + C_1 x + C_2,$$

where $M(x)$ is the internal moment about the y-direction and $H(\dots)$ is the Heaviside step function. The beam is free at the right end. Therefore, the boundary conditions are $Q(a) = 0$ and $M(a) = 0$. The first condition means

$$-FH\left(\frac{a}{2}\right) + C_1 = 0 \implies C_1 = F,$$

since $H(a/2) = 1$ for $a > 0$. Using this result for C_1 in the condition $M(a) = 0$, we can write

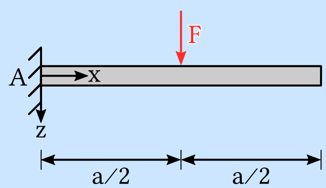
$$F\left(a - \frac{a}{2}\right) + Fa + C_2 = 0 \implies C_2 = -F\frac{a}{2},$$

thus

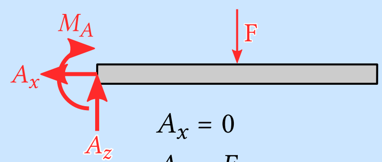
$$Q(x) = F\left[-H\left(x - \frac{a}{2}\right) + 1\right],$$
$$M(x) = F\left[-\left(x - \frac{a}{2}\right) H\left(x - \frac{a}{2}\right) + x - \frac{a}{2}\right].$$

You can verify the solution by computing $Q(x)$ and $M(x)$ from the reaction forces:

Alternative solution using reaction forces

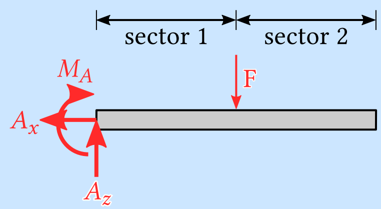


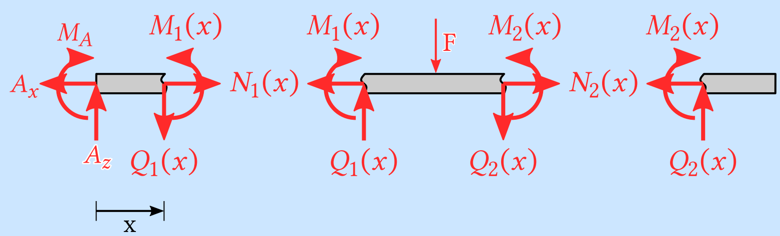
first step: compute the reaction forces at the support from equilibrium of the whole structure



$$A_x = 0$$
$$A_z = F$$
$$M_A = -F\frac{a}{2}$$

second step: divide beam into sectors and compute internal forces





From equilibrium of left side of cut in sector 1:

$$N_1(x) = A_x = 0$$
$$Q_1(x) = A_z = F$$
$$M_1(x) = M_A + A_z x = F\left(x - \frac{a}{2}\right)$$

From quilibrium of right side of cut in sector 2:

$$N_2(x) = Q_2(x) = M_2(x) = 0$$