Exercise 4: Statically determinate beams

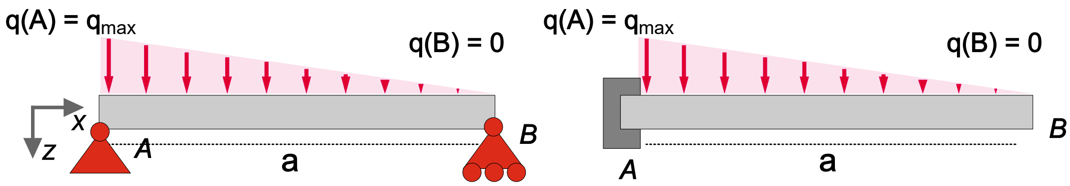
15.11.2024 - 18.11.2024

Question 1 . . . .

Beam AB is subjected to distributed load $q(x)$ that linearly varies along the beam from the maximal value of $q(A) = q_{max}$ on the left end to $q(B) = 0$ on the right. Consider two possible cases to fix the beam (left and right)

(a) Determine internal force $Q(x)$ and internal moment $M(x)$ in the beam.

(b) Find reaction forces and moments on the bearings.



Solution: Both cases are statically determined. This means we can proceed with the integration of the line load to determine internal forces and moments. For both beams the line load is $q(x) = \frac{q_{max}}{a}(a - x)$ and is positive as it points towards the z-direction. For brevity we write $\frac{q_{max}}{a} = q_0$.

(a) Let's determine internal forces and moments.

$$Q(x) = -\int q(x)dx = q_0 \left(\frac{x^2}{2} - ax + C_1 \right),$$
$$M(x) = \int Q(x)dx = q_0 \left(\frac{x^3}{6} - a\frac{x^2}{2} + C_1 x + C_2 \right)$$

In order to determine the integration constants we need to look at the boundary conditions (BCs). The beam on the left is supported by a hinge in A and a roller in B. BCs are:

$$M(0) = 0, M(a) = 0$$

therefore:

$$M(0) = 0 = q_0 C_2 \implies C_2 = 0,$$
$$M(a) = 0 = q_0 \left(\frac{a^3}{6} - \frac{a^3}{2} + C_1 a \right) \implies C_1 = \frac{a^2}{3}$$

The beam on the right is fixed in A and has free end in B. BCs are:

$$Q(a) = 0, M(a) = 0$$

therefore:

$$Q(a) = 0 = q_0 \left(\frac{a^2}{2} - a^2 + C_1 \right) \implies C_1 = \frac{a^2}{2},$$
$$M(a) = 0 = q_0 \left(\frac{a^3}{6} - \frac{a^3}{2} + \frac{a^3}{2} + C_2 \right) \implies C_2 = -\frac{a^3}{6}$$

(b) Let's know find the reactions at the bearings. The beam on the left has a vertical reaction in A and in B:

$$Q(0) = R_A, Q(a) = R_B$$

therefore:

$$R_A = \frac{q_0 a^2}{3},$$
$$R_B = q_0 \left(\frac{a^2}{2} - a^2 + \frac{a^2}{3} \right) = -\frac{q_0 a^2}{6}$$

By looking at the beam load, we expect both reaction forces to point upwards to balance the line load. R_A is positive and at the left-hand side of the beam, this means that this force points upwards. R_B is negative and at the right-hand side of the beam, this means that also this force points upwards.

The beam on the right has a vertical reaction and a moment in A:

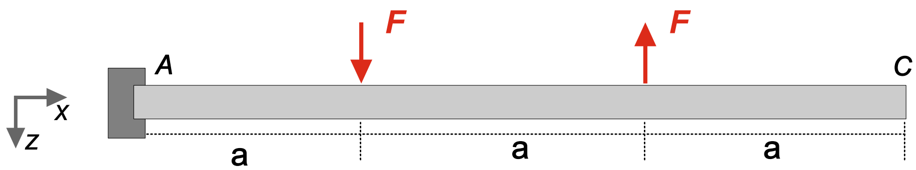
$$Q(0) = R_A, M(0) = M_A$$

therefore:

$$R_A = \frac{q_0 a^2}{2},$$
$$M_A = -\frac{q_0 a^3}{6}$$

By looking at the beam load, we expect the vertical reaction to point upwards and the moment to be counterclockwise. R_A is positive and at the left-hand side of the beam, this means that this force points upwards. M_a is negative and at the left-hand side of the beam, this means that the moment is counterclockwise.

Question 2 . . . .
A beam of length $3a$ is mounted on the wall from the left. Two forces with the same magnitude but different directions are acting on the beam. Please find the internal force $Q(x)$ and moment $M(x)$. *Hint! You can use any method here*



Solution: As the two applied forces are already balancing each other, the easiest method is to calculate the reaction forces and apply the equilibrium equation to the beam.

$$\downarrow \quad R_A + F - F = 0 \implies R_A = 0,$$
$$\text{Ⓐ} \quad M_A - aF + 2aF = 0 \implies M_A = -aF$$

Since there is no vertical reaction in A, the beam has a shear force only in the section between from a to $2a$. According to the sign convention, the internal shear force is negative. We can then easily write:

$$Q(x) = \begin{cases} 0 & 0 < x < a \\ -F & a < x < 2a \\ 0 & 2a < x < 3a \end{cases}$$

$M_A = -aF$ means the reaction moment is clockwise. According to the sign convention, a clockwise moment on the left-end side of the beam is positive. Thus, we can immediately see that in the first third of the beam the internal moment is constant and equal to aF, because shear forces are zero. In the second third of the beam, there is a constant negative shear force, which means the moment will decrease till 0, because there is no moment applied to the free end. We can write:

$$M(x) = \begin{cases} aF & 0 < x < a \\ F(2a - x) & a < x < 2a \\ 0 & 2a < x < 3a \end{cases}$$

We can arrive to the same conclusion by integrating the line load $q(x) = F\delta(x-a) - F\delta(x-2a)$ and knowing the BCs, $Q(3a) = 0$ and $M(3a) = 0$.

$$Q(x) = -F\theta(x-a) + F\theta(x-2a) + C_1$$
$$M(x) = -F(x-a)\theta(x-a) + F(x-2a)\theta(x-2a) + C_1x + C_2$$

with the following BCs:

$$Q(3a) = 0, M(3a) = 0$$

we calculate:

$$Q(3a) = 0 = -F + F + C_1 \implies C_1 = 0,$$
$$M(3a) = 0 = -2aF + aF + C_2 \implies C_2 = aF$$

therefore:

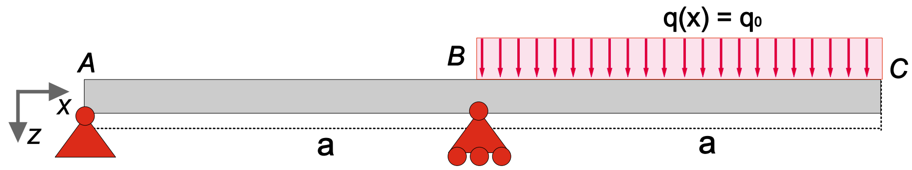
$$Q(x) = -F\theta(x-a) + F\theta(x-2a)$$
$$M(x) = -F(x-a)\theta(x-a) + F(x-2a)\theta(x-2a) + aF$$

Question 3

The beam AC is subjected to line load at the right half. Simultaneously, exactly in the middle, this beam is supported by a roller. If you rotate this image 90 degrees counter-clockwise, you can imagine that this is a high building subjected to wind load at the top floors.

(a) Determine internal force $Q(x)$ and internal moment $M(x)$. At which points do they reach maximum?

(b) Find reactions (forces and moments) on bearings.



Solution:

(a) Note that we have three boundary conditions

$$M(0) = 0$$
$$M(2a) = 0$$
$$Q(2a) = 0.$$

This is because we cannot obtain the vertical reaction of the roller from the internal forces. We therefore need a third condition to determine the reaction force. We can consider the reaction force as an unknown force R_B acting at $x = a$. If we select R_B as pointing upwards (negative z-direction), the line load equation is then the following:

$$q(x) = q_0\theta(x-a) - R_B\delta(x-a)$$

by proceeding with the double integrations we obtain:

$$Q(x) = [-q_0(x-a) + R_B]\theta(x-a) + C_1$$
$$M(x) = [-q_0\frac{(x-a)^2}{2} + R_B(x-a)]\theta(x-a) + C_1x + C_2$$

Now we apply the three BCs to solve for the three unknowns, R_B, C_1 and C_2:

$$M(0) = 0 = C_2$$
$$Q(2a) = 0 = -aq_0 + R_B + C_1 \implies \qquad C_1 = aq_0 - R_B$$
$$M(2a) = 0 = -q_0\frac{a^2}{2} + aR_B + 2q_0a^2 - 2aR_B \implies \qquad R_B = \frac{3}{2}q_0a$$
$$C_1 = -\frac{q_0a}{2}$$

R_B has a positive sign, this means that the force is actually pointing upwards.

(b) We already found the reaction of the roller in the previous part. We have to find the reaction of the hinge, R_A:

$$R_A = Q(0) = -\frac{q_0a}{2}$$

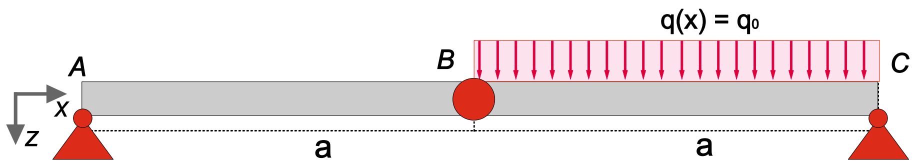
Beacuse the shear force is negative on the left-hand side of the beam, the reaction force points downwards.

Question 4

This is a small modification of Q3. What changes if instead of one beam supported in the middle, we consider two separate beams connected by a hinge? Note that the rightmost hinge has been "upgraded" from roller to proper hinge. Do you understand why?

(a) Answer the same questions as in Q3

(b) Compare your results with Q3.



Solution: This beam is not statically determined as the three hinges are aligned and the structure can have an infinitesimal vertical displacement in point B!