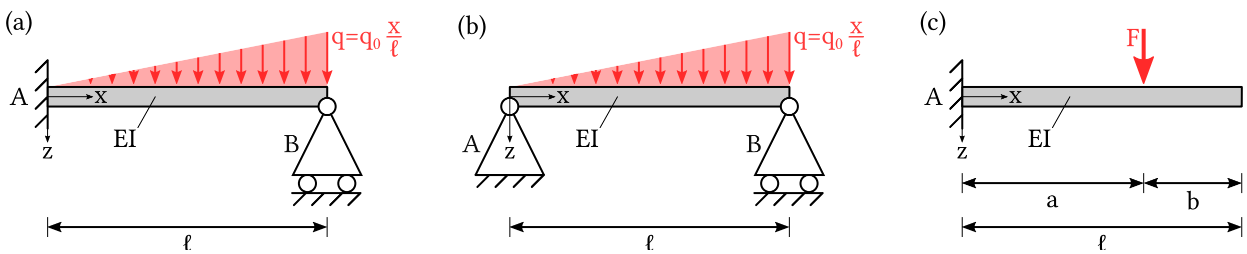
Exercise 5: Bending

22.11.2024 - 25.11.2024

Question 1 . . . .

Determine if the following structures are statically determinate! Calculate the deflection and the reaction forces and moments at the supports! Sketch the shear force, moment and deflection.



Solution: Recall that a structure is statically determinate if

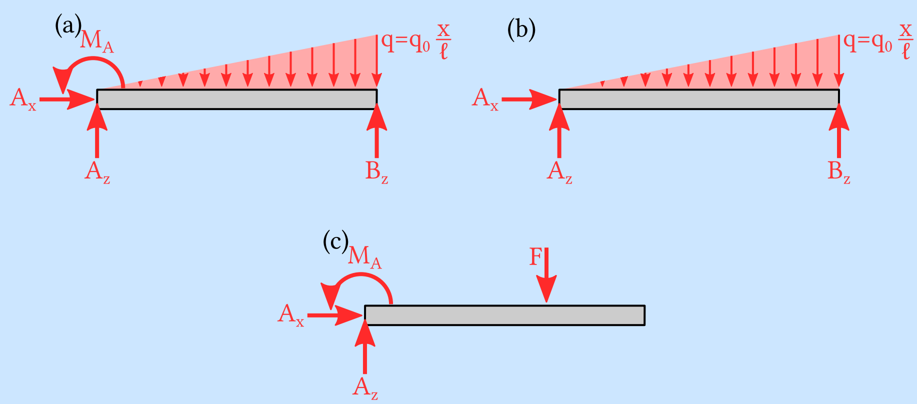
$$3n - (r + v) = 0,$$

where n is the number of bodies, r the number of reaction forces or moments of the supports, and v the number of forces or moments transmitted at links.

(a) $n = 1$, $r = 4$, $v = 0 \to$ indeterminate / overconstrained

(b) $n = 1$, $r = 3$, $v = 0 \to$ determinate

(c) $n = 1$, $r = 3$, $v = 0 \to$ determinate



In order to calculate the z-deflection $w(x)$ of the beam we have to integrate the Euler-Bernoulli equation. However, there are two possible starting points. Either we integrate $EIw''''(x)$ four times, or we first determine the bending moment as a function of position ($M(x)$) and then integrate $EIw''(x) = -M(x)$ two times. In both cases we need to make use of the boundary conditions in order to determine the constants of integration. However, in the second case, there will be fewer constants of integration.

Since problem (a) is indeterminate we cannot immediately calculate $M(x)$ and therefore need to follow the first approach. Note that problems (a) and (b) have the same geometry, load and boundary condition on the right hand side. Hence, (b) can be solved quickly by recycling the solution of (a).

(a)

$$
\begin{aligned}
EIw''''(x) &= q_0 \frac{x}{l} \\
EIw'''(x) &= \frac{1}{2} q_0 \frac{x^2}{l} + C_1 \\
EIw''(x) &= \frac{1}{6} q_0 \frac{x^3}{l} + C_1 x + C_2 \\
EIw'(x) &= \frac{1}{24} q_0 \frac{x^4}{l} + \frac{1}{2} C_1 x^2 + C_2 x + C_3 \\
EIw(x) &= \frac{1}{120} q_0 \frac{x^5}{l} + \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4
\end{aligned}
$$

Now we apply boundary conditions:

$$
w(x=0) = 0 \implies C_4 = 0,
$$

$$
w'(x=0) = 0 \implies C_3 = 0,
$$

$$
\left.\begin{aligned} & w''(x=l) = 0 \quad \text{(zero moment)} \\ & w(x=l) = 0 \end{aligned}\right\} \implies C_1 = -\frac{9}{40} q_0 l, \quad C_2 = \frac{7}{120} q_0 l^2.
$$

The solution for the displacements can thus be written as

$$
w(x) = \frac{1}{EI} \left(\frac{1}{120} q_0 \frac{x^5}{l} - \frac{3}{80} q_0 l x^3 + \frac{7}{240} q_0 l^3 x \right).
$$

The reaction forces and moments can be obtained by evaluating the appropriate derivatives of $w(x)$ at the location of the bearings. To get the correct sign, draw the reaction forces and moments with an arbitrary sense. Then imagine a cut and require that the bearing force/moment and the reaction force/moment sum to zero.

$$
\begin{aligned}
A_z &= Q(x=0) = -EIw'''(x=0) = -C_1 = \frac{9}{40} q_0 l, \\
B_z &= -Q(x=l) = EIw'''(x=l) = \frac{11}{40} q_0 l, \\
M_A &= -M(x=0) = EIw''(x=0) = C_2 = \frac{7}{120} q_0 l^2, \\
A_x &= 0 \quad \text{(equilibrium)}
\end{aligned}
$$

We could check the solution for A_z, B_z and M_A by checking equilibrium of the whole structure.

(b) The line load is the same in (a), therefore the leading term of $w(x)$ is $q_0 x^5/120l$, as before. The boundary conditions are

$$
w(x=0) = 0 \implies C_4 = 0, \quad w''(x=0) = 0 \implies C_2 = 0, \quad w''(x=l) = 0 \quad \text{(zero moment)} \implies C_1 = -\frac{1}{6} q_0 l
$$

$$
w(x=l) = 0 \implies C_3 = \frac{7}{360} q_0 l^3,
$$

hence the solution for the displacements is

$$
w(x) = \frac{1}{EI} \left(\frac{1}{120} q_0 \frac{x^5}{l} - \frac{1}{36} q_0 l x^3 + \frac{7}{360} q_0 l^3 x \right).
$$

The reactions forces are

$$
\begin{aligned}
A_z &= Q(x=0) = -EIw'''(x=0) = -C_1 = \frac{1}{6} q_0 l, \\
B_z &= -Q(x=l) = EIw'''(x=l) = \frac{1}{3} q_0 l.
\end{aligned}
$$

The same solution should be obtained by consideration of equilibrium of the whole structure.

(c) The problem is statically determinate, hence the reactions at the support can be obtained by requiring equilibrium of the whole structure,

$$
\begin{aligned}
A_x &= 0, \\
A_z &= F, \\
M_A &= Fa.
\end{aligned}
$$

We divide the structure into two sectors, $0 \leq x \leq a$ (sector 1) and $a \leq x \leq l$ (sector 2). The internal moment is

$$
\begin{aligned}
M^{(1)}(x) &= F\left(x - a\right), \\
M^{(2)}(x) &= 0.
\end{aligned}
$$

The deflection is obtained by integrating the second derivative of $w''(x)$.

$$
\begin{aligned}
EIw^{(1)''}(x) &= F\left(x - a\right), \\
EIw^{(1)'}(x) &= -\frac{1}{2}Fx^2 + Fax + C_1, \\
EIw^{(1)}(x) &= -\frac{1}{6}Fx^3 + \frac{1}{2}Fax^2 + C_1 x + C_2, \\
EIw^{(2)''}(x) &= 0, \\
EIw^{(2)'}(x) &= C_3, \\
EIw^{(2)}(x) &= C_3 x + C_4.
\end{aligned}
$$

Consideration of the boundary conditions gives the solution for the constants of integration,

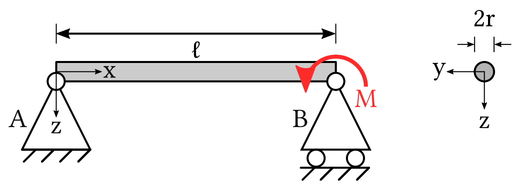
$$
\begin{aligned}
w^{(1)'}(x=0) = 0 &\implies C_1 = 0, \\
w^{(1)}(x=0) = 0 &\implies C_2 = 0, \\
w^{(1)'}(x=a) = w^{(2)'}(x=a) &\implies C_3 = \frac{1}{2}Fa^2, \\
w^{(1)}(x=a) = w^{(2)}(x=a) &\implies C_4 = -\frac{1}{6}Fa^3.
\end{aligned}
$$

The solution for the deflection is therefore

$$
\begin{aligned}
w^{(1)}(x) &= \frac{1}{EI}\left[-\frac{1}{6}Fx^3 + \frac{1}{2}Fax^2\right], \\
w^{(2)}(x) &= \frac{1}{EI}\left[\frac{1}{2}Fa^2x - \frac{1}{6}Fa^3\right].
\end{aligned}
$$

Question 2 . . . .

A beam with cylindrical cross-section (radius r) is supported by two bearings, see below. A moment M is applied at one end. The area moment of inertia for this cross-section is $I = \pi r^4/4$. Calculate the maximum deflection! Where does it occur? Sketch the shear force, moment and deflection.

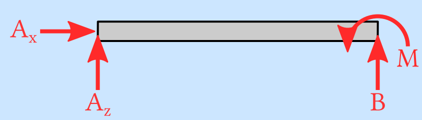


Solution: *Reference*:Gross, Hauger, Schröder, Wall, Technische Mechanik 2, 9th edition, Springer Vieweg (pages 122–123).

Recall that a structure is statically determinate if

$$3n - (r + v) = 0,$$

where n is the number of bodies, r the number of reaction forces or moments of the supports, and v the number of forces or moments transmitted at links. Here $n = 1$, $r = 3$, $v = 0 \to$ the structure is statically determinate.



From equilibrium, we have $A_x = 0$ and $A_z = -B = -M/l$. The internal moment is

$$M(x) = -xA_z = M\frac{x}{l}.$$

Let E be Young's modulus and I_y the second moment of area for bending about y. Integration of the differential equation of the bending line yields

$$\begin{aligned} EI_y w'' &= -\frac{M}{l}x \\ EI_y w' &= -\frac{M}{2l}x^2 + C_1 \\ EI_y w &= -\frac{M}{6l}x^3 + C_1 x + C_2 \end{aligned}$$

The boundary conditions are $w(0) = 0$ and $w(l) = 0$. Inserting into the last equation gives $C_2 = 0$ and $C_1 = \frac{Ml}{6}$. Thus, we have

$$w(x) = \frac{1}{EI_y}\left(-\frac{M}{6l}x^3 + \frac{Ml}{6}x\right).$$

The maximum value of w occurs at the position x^* where $w'(x^*) = 0$, i.e.

$$-\frac{M}{2l}(x^*)^2 + \frac{Ml}{6} = 0 \to x^* = \frac{1}{\sqrt{3}}l.$$

Thus

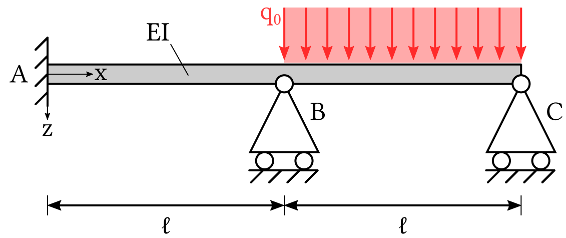
$$w(x^*) = \frac{\sqrt{3}Ml^2}{27EI_y}.$$

For the circular cross-section, we have from exercise 1(b) $I_y = \frac{\pi}{4}r^4$. Inserting gives

$$w(x^*) = \frac{4\sqrt{3}Ml^2}{27\pi E r^4}.$$

Question 3 . . . .

The beam shown below has the bending stiffness EI and is subjected to a line load q_0. Calculate the reaction forces and the deflection of the beam! Sketch the shear force, moment and deflection.



Hint: If a system is hyperstatic it might be helpful to start from the Euler-Bernoulli equation before trying to determine the reaction forces.

Solution: This structure is composed of one element ($r = 1$) and four bearings, which create five reactions ($r = 5$). Testing for determinacy, we find

$$3n - (r + v) = -2, \tag{1}$$

i.e. the structure is hyperstatic. We cannot find all reactions by consideration of equilibrium alone. Thus, we will first solve the Euler-Bernoulli equation and then obtain the reactions from the solution.

There is a discontinuity at the support B, hence we need to find separate solutions for the two sectors $0 \le x \le \ell$ (sector 1) and $\ell \le x \le 2\ell$ (sector 2). Let $w^{(1)}(x)$ be the deflection along z in sector 1. There is no line load, hence

$$EI{w^{(1)}}''''(x) = 0, \tag{2}$$
$$EI{w^{(1)}}'''(x) = C_1^{(1)}, \tag{3}$$
$$EI{w^{(1)}}''(x) = C_1^{(1)}x + C_2^{(1)}, \tag{4}$$
$$EI{w^{(1)}}'(x) = \frac{1}{2}C_1^{(1)}x^2 + C_2^{(1)}x + C_3^{(1)}, \tag{5}$$
$$EIw^{(1)}(x) = \frac{1}{6}C_1^{(1)}x^3 + \frac{1}{2}C_2^{(1)}x^2 + C_3^{(1)}x + C_4^{(1)}, \tag{6}$$

where $C_1^{(1)}$, $C_2^{(1)}$, $C_3^{(1)}$, and $C_4^{(1)}$ are constants of integration.

In sector 2, the line load is q_0, therefore

$$EI{w^{(2)}}''''(x) = q_0, \tag{7}$$
$$EI{w^{(2)}}'''(x) = q_0x + C_1^{(2)}, \tag{8}$$
$$EI{w^{(2)}}''(x) = \frac{1}{2}q_0x^2 + C_1^{(2)}x + C_2^{(2)}, \tag{9}$$
$$EI{w^{(2)}}'(x) = \frac{1}{6}q_0x^3 + \frac{1}{2}C_1^{(2)}x^2 + C_2^{(2)}x + C_3^{(2)}, \tag{10}$$
$$EIw^{(2)}(x) = \frac{1}{24}q_0x^4 + \frac{1}{6}C_1^{(2)}x^3 + \frac{1}{2}C_2^{(2)}x^2 + C_3^{(2)}x + C_4^{(2)}, \tag{11}$$

where $C_1^{(2)}$, $C_2^{(2)}$, $C_3^{(2)}$, and $C_4^{(2)}$ are constants of integration.

The following boundary conditions apply:

$$
\begin{aligned}
w^{(1)}(x=0) &= 0 \quad \text{(beam is clamped)},\\
w^{(1)}(x=0)' &= 0 \quad \text{(beam is clamped)},\\
w^{(1)}(x=l) = w^{(2)}(x=l) &= 0, \quad \text{(support at } B\text{)},\\
w^{(1)\prime}(x=l) &= w^{(2)\prime}(x=l), \quad \text{(no kink at } B\text{)},\\
w^{(1)\prime\prime}(x=l) &= w^{(2)\prime\prime}(x=l), \quad \text{(moment continuous at } B\text{)},\\
w^{(2)}(x=2l) &= 0 \quad \text{(support at } C\text{)}.\\
w^{(2)\prime\prime}(x=2l) &= 0 \quad \text{(no moment at support } C\text{)}.
\end{aligned} \tag{12}
$$

By using the first two boundary conditions, we find $C_4^{(1)} = C_3^{(1)} = 0$. Inserting the third boundary condition in Eq. 6, we get

$$
C_2^{(1)} = -\frac{1}{3} C_1^{(1)} l. \tag{13}
$$

$w^{(2)}(x=l) = 0$ and $w^{(2)}(x=2l) = 0$ imply

$$
C_4^{(2)} = -\left(\frac{1}{24} q_0 l^4 + \frac{1}{6} C_1^{(2)} l^3 + \frac{1}{2} C_2^{(2)} l^2 + C_3^{(2)} l\right), \tag{14}
$$

$$
C_4^{(2)} = -\left(\frac{2}{3} q_0 l^4 + \frac{4}{3} C_1^{(2)} + 2C_2^{(2)} l^2 + 2C_3^{(2)} l\right), \tag{15}
$$

which can be combined to give

$$
C_3^{(2)} = -\left(\frac{5}{8} q_0 l^3 + \frac{7}{6} C_1^{(2)} l^2 + \frac{3}{2} C_2^{(2)} l\right), \tag{16}
$$

$$
C_4^{(2)} = \frac{7}{12} q_0 l^4 + C_1^{(2)} l^3 + C_2^{(2)} l^2. \tag{17}
$$

$w^{(2)\prime\prime}(x=2l) = 0$ yields

$$
C_2^{(2)} = -2 q_0 l^2 - 2 C_1^{(2)} l.
$$

The only remaining unknowns are now $C_1^{(1)}$ and $C_1^{(2)}$. Thus far, we have

$$
\begin{aligned}
EIw^{(1)}(x) &= \frac{1}{6} C_1^{(1)} \left(x^3 - l x^2\right),\\
EIw^{(2)}(x) &= -\frac{1}{24} q_0 x^4 + \frac{1}{6} C_1^{(2)} x^3 - \left(q_0 l + C_1^{(2)}\right) l x^2 + \frac{19}{8} q_0 l^3 x + \frac{11}{6} C_1^{(2)} l^2 x - \frac{17}{12} q_0 l^4 - C_1^{(2)} l^3.
\end{aligned}
$$

Finally, the conditions $w^{(1)\prime}(x=l) = w^{(2)\prime}(x=l)$ and $w^{(1)\prime\prime}(x=l) = w^{(2)\prime\prime}(x=l)$ at B yield

$$
\begin{aligned}
C_1^{(1)} &= \frac{3}{28} q_0 l,\\
C_1^{(2)} &= -\frac{11}{7} q_0 l.
\end{aligned}
$$

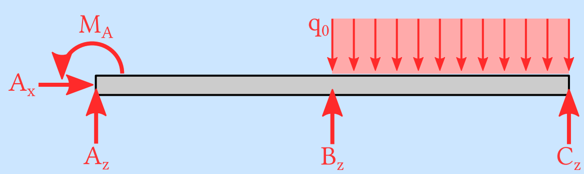
The deflections are therefore

$$w^{(1)}(x) = \frac{1}{EI}\left[\frac{1}{56}q_0 l\left(x^3 - lx^2\right)\right],$$
$$w^{(2)}(x) = \frac{1}{EI}\left[-\frac{1}{24}q_0 x^4 - \frac{11}{42}q_0 l x^3 + \frac{4}{7}q_0 l^2 x^2 + \frac{19}{8}q_0 l^3 x - \frac{121}{42}q_0 l^3 x - \frac{17}{12}q_0 l^4 + \frac{11}{7}q_0 l^4\right].$$

The reaction forces can be obtained from the derivatives of the deflections,

$$A_z = -EI{w^{(1)}}'''(x=0) = -\frac{3}{28}q_0 l,$$
$$M_A = -EI{w^{(1)}}''(x=0) = \frac{1}{28}q_0 l^2,$$
$$-C_z = -EI{w^{(2)}}'''(x=2l) \rightarrow C_z = \frac{3}{7}q_0 l.$$

$A_x = 0$ and $B_z = \frac{19}{28}q_0 l$ follow from equilibrium.



A final note: dividing the structure into different sectors and finding separate solutions, as was done here, can be a bit tedious. A shorter and more elegant solution is possible using MACAULAY brackets (FÖPPEL brackets), see Hauger, Lippmann, Mannl, Werner, Aufgaben zur Technischen Mechanik 1–3, 3d ed. Springer (p. 224–225).