Exercise 8: Hooke's law

13.12.2024 - 16.12.2024

Question 1 . . . .

Reference: Chou, Pagano, *Elasticity: Tensor, Dyadic, and Engineering Approaches*, Dover Publications, p. 64.

Determine the slope of the σ_{xx} vs. ε_{xx} curve in the elastic range if a material is tested under the following state of stress:

$$\sigma_{xx} = 2\sigma_{yy} = 3\sigma_{zz}$$
$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$$

Solution: Recall Hooke's law:

$$\begin{aligned}
\sigma_{xx} &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right), \\
\sigma_{yy} &= 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right), \\
\sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right), \\
\sigma_{xy} &= 2\mu\varepsilon_{xy}, \\
\sigma_{xz} &= 2\mu\varepsilon_{xz}, \\
\sigma_{yz} &= 2\mu\varepsilon_{yz}.
\end{aligned}$$

We see immediately that $\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$. The first equation gives

$$\lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = \sigma_{xx} - 2\mu\varepsilon_{xx}.$$

Replacing this term in the second and third equations allows to determine ε_{yy} and ε_{zz} as function of σ_{xx} and ε_{xx},

$$\begin{aligned}
\varepsilon_{yy} &= \varepsilon_{xx} - \frac{\sigma_{xx}}{4\mu}, \\
\varepsilon_{zz} &= \varepsilon_{xx} - \frac{\sigma_{xx}}{3\mu}.
\end{aligned}$$

Inserting back into the first equation allows to eliminate ε_{yy} and ε_{zz}. Thus, we can write σ_{xx} as a function of ε_{xx} alone,

$$\sigma_{xx} = \frac{12\mu\left(3\lambda + 2\mu\right)}{\left(12\mu + 7\lambda\right)}\varepsilon_{xx}$$

and read off the slope

$$\frac{\partial\sigma_{xx}}{\partial\varepsilon_{xx}} = \frac{12\mu\left(3\lambda + 2\mu\right)}{\left(12\mu + 7\lambda\right)}.$$

Question 2 . . . .

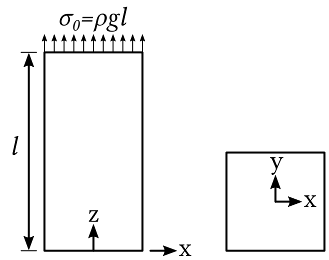
Reference: Chou, Pagano, *Elasticity: Tensor, Dyadic, and Engineering Approaches*, Dover Publications, p. 86.

A bar of constant mass density ρ hangs under its own weight and is supported by the uniform stress σ_0 as shown in the figure. Assume that the stresses σ_{xx}, σ_{yy}, σ_{xy}, σ_{xz}, and σ_{yz} vanish identically.

(a) Recall that there are 15 governing equations in $3D$: three equilibrium equations, six strain-displacement relations, and six stress-strain relations. Show that the 15 equations reduce to seven equations under the assumptions above. What are the variables?

(b) Integrate the equilibrium equation to show that $\sigma_{zz} = \rho g z$, where g is the acceleration due to gravity. Also show that the prescribed boundary conditions are satisfied by this solution.

(c) Find ε_{xx}, ε_{yy}, and ε_{zz} from Hooke's law.



Solution: (a) The governing equation for isotropic elasticity in 3D are

$$\left.\begin{aligned}\sigma_{xx} &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right)\\ \sigma_{yy} &= 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right)\\ \sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right)\\ \sigma_{xy} &= 2\mu\varepsilon_{xy}\\ \sigma_{xz} &= 2\mu\varepsilon_{xz}\\ \sigma_{yz} &= 2\mu\varepsilon_{yz}\end{aligned}\right\}\text{Hooke's law,}$$

$$\left.\begin{aligned}\varepsilon_{xx} &= \frac{\partial u_x}{\partial x}\\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y}\\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y}\\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\\ \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)\\ \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)\end{aligned}\right\}\text{strain-displacement relations,}$$

$$\left.\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+F_x &= 0\\ \frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+F_y &= 0\\ \frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+F_z &= 0\end{aligned}\right\}\text{equilibrium}$$

Equations colored red disappear. If $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$ and $\mu \neq 0$, then the three red equations in Hooke's law can only be fulfilled if $\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$. All three become $0 = 0$ and can therefore be ignored. The left hand side of the three red strain-displacement relations is zero. They can only be fulfilled if (i) the derivatives on the right hand side are zero, or (ii) if $\partial u_x/\partial y = -\partial u_y/\partial x$, $\partial u_x/\partial z = -\partial u_z/\partial x$, and $\partial u_y/\partial z = -\partial u_z/\partial y$. Case (ii) represents a state of superimposed rigid body rotation, which we can ignore. Two equilibrium equations disappear because the corresponding stresses are zero. Note that gravitation acts only along z, hence $F_x = F_y = 0$. The remaining variables are σ_{zz}, u_x, u_y, u_z, ε_{xx}, ε_{yy}, and ε_{zz}. Note that there are now seven governing equations and seven variables.

Solution: (b) The gravitation force per volume is $F_z = -\rho g$. The remaining equilibrium equation reads

$$\frac{\partial \sigma_{zz}}{\partial z} - \rho g = 0,$$

which can be integrated to give

$$\sigma_{zz} = \rho g z + C.$$

C is a constant. σ_{zz} must vanish at $z = 0$, hence $C = 0$.

Solution: (c) Hooke's law now reads

$$\begin{aligned} 0 &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right), \\ 0 &= 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right), \\ \sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right). \end{aligned}$$

By subtracting the first and the second equation, we find $\varepsilon_{xx} = \varepsilon_{yy}$. Setting $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{\text{lateral}}$ and inserting back into the first or second equation, we then obtain

$$\varepsilon_{\text{lateral}} = -\frac{\lambda}{2\left(\lambda + \mu\right)}\varepsilon_{zz}.$$

Note that $\frac{\lambda}{2(\lambda+\mu)} \equiv \nu$, the Poisson constant. Inserting $\varepsilon_{\text{lateral}}$ and the result for σ_{zz} from (b) into the last equation, we get

$$\varepsilon_{xx} = \left(2\mu + \lambda - \frac{\lambda^2}{\lambda + \mu}\right)^{-1} \rho g z.$$

Question 3 . . . .

Metal or semiconductor crystals may contain defects in their lattice structure called "dislocations". These are very important for understanding plastic deformation. A so-called "screw dislocation", sketched in the figure, is created by the following displacement

$$\mathbf{u}(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ \frac{b}{2\pi}\arctan\left(\frac{y}{x}\right) \end{bmatrix}.$$

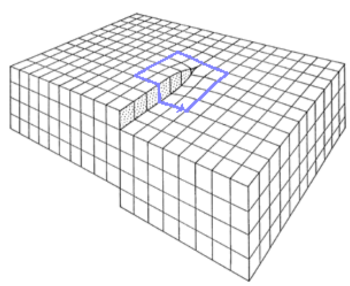


Figure 1: screw dislocation from: `https://www.tf.uni-kiel.de/matwis/amat/defen/kap5/backbone/r522.html`

Calculate the associated strain tensor ε and the stress tensor σ (using Hooke's law)! Is the body in a state of plane strain or plane stress? Do you notice something peculiar near the center of the dislocation at $x = y = 0$?

Solution: The strains are given by the equation

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$$

we thus find

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yx} = 0, \\
\varepsilon_{xz} &= \varepsilon_{zx} = -\frac{b}{4\pi}\frac{y}{x^2+y^2}, \\
\varepsilon_{yz} &= \varepsilon_{zy} = \frac{b}{4\pi}\frac{x}{x^2+y^2},
\end{aligned}$$

and the stresses can be computed by the formula for isotropic materials given in the lecture

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

to find

$$\begin{aligned}
\sigma_{xx} &= \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yx} = 0, \\
\sigma_{xz} &= \sigma_{zx} = -\frac{\mu b}{2\pi}\frac{y}{x^2+y^2}, \\
\sigma_{yz} &= \sigma_{zy} = \frac{\mu b}{2\pi}\frac{x}{x^2+y^2}.
\end{aligned}$$

The state of deformation is neither plane strain nor plane stress. Note that the fields diverge as $x, y \to 0$. Thus small strain elasticity breaks down in some region around $x = y = 0$ and one needs to consider the atomic structure of the material to find the true state of deformation.

Question 4 . . . .

We now consider a state of plane strain. The governing equations are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \quad \text{(definition of strain)},$$

$$\sigma_{xx} = 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}\right), \quad \sigma_{yy} = 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}\right), \quad \sigma_{xy} = 2\mu\varepsilon_{xy} \quad \text{(Hooke's law)},$$

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + F_x = 0, \quad \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x} + F_y = 0, \quad \text{(equilibrium)}.$$

These are eight governing equations. However, we can combine them in such a way that we end up with only two equations in terms of the displacement components u_x and u_y. This form is convenient for problems where displacement components are prescribed over the entire boundary of the body. Find these two equations!

Solution: By subsituting the strains into Hooke's law, one obtains

$$\begin{aligned}
\sigma_{xx} &= \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + 2\mu\frac{\partial u_x}{\partial x}, \\
\sigma_{yy} &= \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + 2\mu\frac{\partial u_y}{\partial y}, \\
\sigma_{xy} &= \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right).
\end{aligned}$$

Inserting these equations into the equilibrium conditions and eliminating stresses gives

$$\begin{aligned}
\mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + F_x = 0, \\
\mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + F_y = 0.
\end{aligned}$$

These are the *Navier-Cauchy* equations for plane strain.