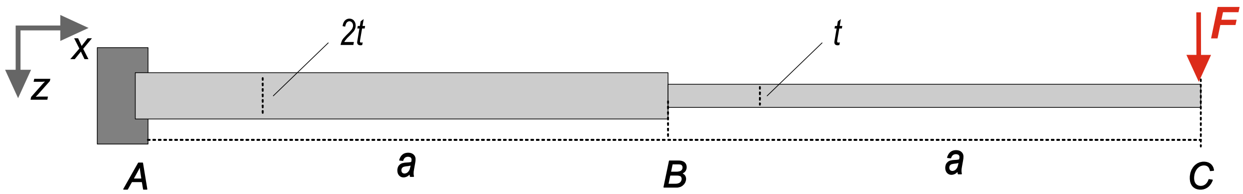
Exercise 9: Beams again

10.01.2025 - 13.01.2025

Question 1 . . . .

A horizontal beam AC with the total length of $2a$ consists of two rectangular sections with different thickness as shown in Figure. The left section AB of length a and thickness $2t$ is attached to the wall on the left. Force F is applied on the rightmost point section BC of length a and thickness t. Out-of-plane width of both sections is w and both sections are made of the material with Young's modulus E.



(a) Is the problem statically determinate?

(b) How far from point A is the center of mass of the entire system located?

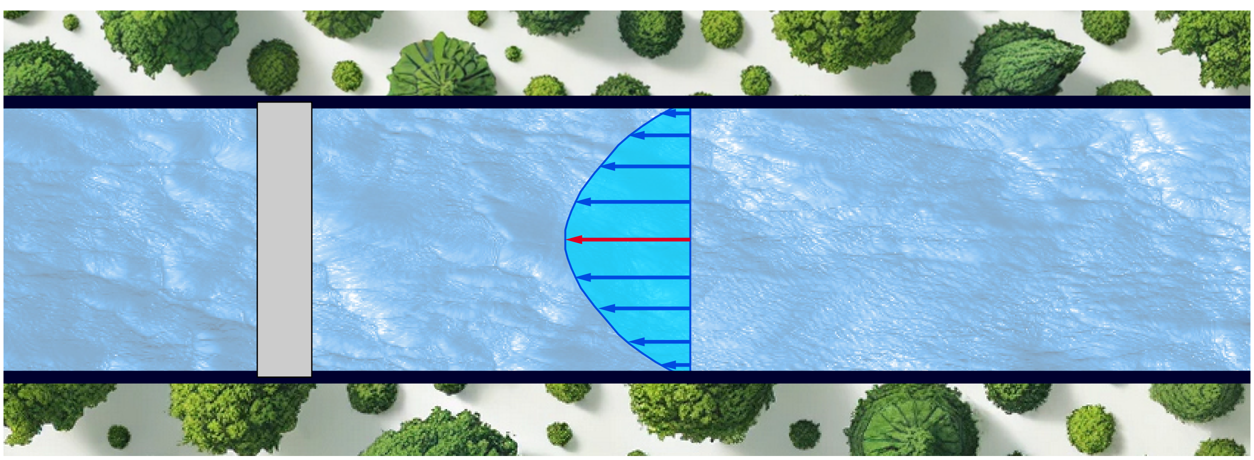
(c) What are the reactions and moments at point A?

(d) Calculate the internal shear forces $Q(x)$ and moments $M(x)$.

(e) Calculate the deflection profile (bending line) $w(x)$ of the beam. *Note! Integration constants are expected to be found, but failing to do so will not result in a large point deduction.*

Question 2 . . . .

A channel of width $2a$ with a right-to-left current is blocked by a permeable floating bridge (beam). From fluid dynamics, we know that the speed of the current changes across the channel according to a quadratic law, with the maximum speed v_{max} at the center of the channel (red arrow) and $v_{\text{min}} = 0$ next to the banks.



(a) Write down the expression for $v(x)$ to satisfy the problem formulation.

(b) From fluid dynamics, we know that at each point, the force $q(x)$ exerted on the net is proportional to the square of the flow speed at that point, $v(x)$. So, $q(x) = \alpha v(x)^2$, where α is a coefficient (dependent on fluid density and drag coefficient, but not important here). Write down the expression for the line load $q(x)$ exerted on the beam.

(c) Calculate the total load acting on the beam.

(d) The ends of the beam are fully constrained. Is the problem statically determinate?

(e) Find the deflection of the beam due to the current, assuming the beam is made of a material with Young's modulus E and its cross-section has a second moment of inertia I. *Note! You do not need to solve for the integration constants, but provide the final expression for $w(x)$ with the correct boundary conditions.*

Question 3 . . . .

To celebrate Spring Equinox that will happen in a couple of weeks, group of astronomers decided to decorate their lab with some astronomical memorabilia. To recreate exponential curve that is very important for astronomy, they hanged a lot of thin steel wires with various lengths $l(x)$ on a plastic curtain rod with tubular cross-section, as shown in figure. However, astronomers were not so good in mechanics, so they completely forgot that plastic rod can deform under the weight of steel wires. Help astronomers to sort things out by solving the following problems.

(a) Calculate the axial moment of inertia I_y for given cross-section with outer diameter d_o and inner diameter d_i.

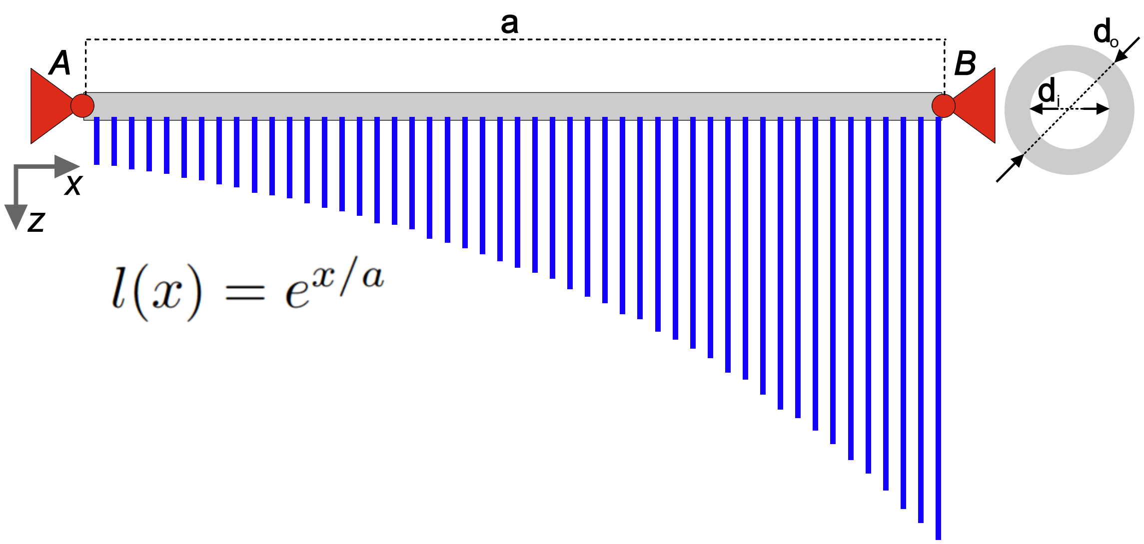
(b) Assuming that the wires are very thin, their lengths can be represented as a function of horizontal coordinate x, namely $l(x) = e^{x/a}$. Denoting the linear density (weight per unit of length) of the steel as ρ, write down the expression for line load acting on horizontal beam AB.

(c) Is the problem statically determinate?

(d) Find deflection of the beam $w(x)$.

(e) Determine reaction forces and moments at the beam ends.

(f) Taking into account the deflection of the beam, do the bottom ends of the steel wires still form an exponential curve?



Solution:

(a) The axial moment of inertia is that of a square bare, but we have to substract the smaller inner bar without material. This gives $I = (b_1^4 - b_2^4)/12 = 507500\ mm^4$.

(b) The prefactor can be determined from the total mass m of the snow on the beam. We have

$$mg = \int_0^a \mathrm{d}x\, q_0 \sin\left(\frac{\pi x}{a}\right) = -\frac{q_0 a}{\pi} \cos\left(\frac{\pi x}{a}\right)\Bigg|_0^a = \frac{2q_0 a}{\pi}, \tag{1}$$

hence $q_0 = \pi mg/(2a)$.

(c) The problem is not statically determine. The 3 degrees-of-freedom of the beam are fixed by 6 constraints.

(d) To find the deflection of the beam, we need to solve the Euler-Bernoulli equations. We find

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q_0 \sin\left(\frac{\pi x}{a}\right) \tag{2}$$

$$EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = -\frac{q_0 a}{\pi}\cos\left(\frac{\pi x}{a}\right) + C_1 \tag{3}$$

$$EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{q_0 a^2}{\pi^2}\sin\left(\frac{\pi x}{a}\right) + C_1 x + C_2 \tag{4}$$

$$EI\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{q_0 a^3}{\pi^3}\cos\left(\frac{\pi x}{a}\right) + \frac{C_1}{2}x^2 + C_2 x + C_3 \tag{5}$$

$$EIw(x) = -\frac{q_0 a^4}{\pi^4}\sin\left(\frac{\pi x}{a}\right) + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4 \tag{6}$$

for the deflection. The boundary conditions

$$w(0) = 0 \tag{7}$$

$$\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=0} = 0 \tag{8}$$

$$w(a) = 0 \tag{9}$$

$$\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=a} = 0 \tag{10}$$

yield

$$C_4 = 0 \tag{11}$$

$$\frac{q_0 a^3}{\pi^3} + C_3 = 0 \tag{12}$$

$$\frac{a^3}{6}C_1 + \frac{a^2}{2}C_2 + aC_3 = 0 \tag{13}$$

$$-\frac{q_0 a^3}{\pi^3} + \frac{a^2}{2}C_1 + aC_2 + C_3 = 0. \tag{14}$$

This can be solved to give

$$C_1 = 0 \tag{15}$$

$$C_2 = \frac{2q_0 a^2}{\pi^3} \tag{16}$$

$$C_3 = -\frac{q_0 a^3}{\pi^3}. \tag{17}$$

The full bending line is hence

$$EIw(x) = \frac{q_0 a^2}{\pi^3}\left[\frac{a^2}{\pi}\sin\left(\frac{\pi x}{a}\right) + x^2 - ax\right]. \tag{18}$$

(e) The shear force $Q(x) = EI\,\mathrm{d}^3 w/\mathrm{d}x^3$ and the moment $M(x) = EI\,\mathrm{d}^2 w/\mathrm{d}x^2$. The reactions forces are the jumps in the shear force. Hence we have $R_A = -Q(0)$ and $R_B = Q(a)$. Evaluation yields $R_A = R_B = q_0 a/\pi$. The reaction moments $M_A = M(0)$ and $M_B = M(a)$. (The sign would require us to define a convention for the rotation of the moment.) This yields $M_A = M_B = 2q_0 a^2/\pi^3$.