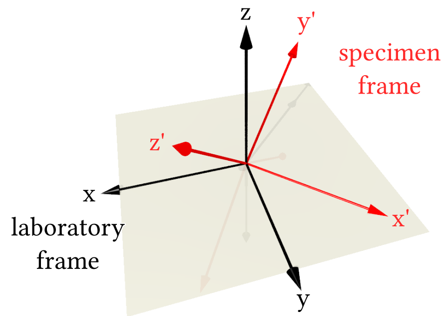
Exercise 10: Stress tensor

17.01.2025 - 20.01.2025

Question 1 . . . .

In this exercise, we will practice tensor rotation. Consider the two coordinate systems in the figure on the right. The red coordinate system ("specimen frame") has been rotated. The basis vectors of this system with respect to the laboratory frame are



$$\mathbf{x}' = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \qquad \mathbf{y}' = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}, \qquad \mathbf{z}' = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Suppose we are given the representation of a stress tensor in the *specimen frame*,

$$\boldsymbol{\sigma}' = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & \sigma_{zz}\end{bmatrix}.$$

This stress tensor would be created by a force that acts on the plane whose normal is $\mathbf{z}'$, along the $\mathbf{z}'$-direction. We want the representation $\boldsymbol{\sigma}$ of this stress tensor in the *laboratory frame*.

(a) Find the rotation matrix $\mathbf{R}$ which, given the representation of a vector in the laboratory frame, yields the representation in the specimen frame upon matrix-vector multiplication!

(b) Verify that the determinant of $\mathbf{R}$ is equal to 1!

(c) Perfom tensor rotation to obtain $\boldsymbol{\sigma}$!

(d) Calculate the von Mises stress for $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}$!

Solution:

(a)
The task is to find the rotation matrix $\mathbf{R}$ which rotates a vector from the laboratory frame (L) into the specimen frame (S), given the axes $(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ of the specimen frame coordinate system. We make this clear by calling this rotation $\mathbf{R}_{L\to S}$. To find the rotation $\mathbf{R}_{L\to S}$ we will first derive the rotation matrix $\mathbf{R}_{cL\to cS}$ for the axes of the coordinate systems. So $\mathbf{R}_{cL\to cS}$ rotates the coordinate system of the laboratory frame into the coordinate system of the specimen frame, i.e.

$$\mathbf{R}_{cL\to cS}\cdot\mathbf{x} = \mathbf{x}' \quad , \qquad \mathbf{R}_{cL\to cS}\cdot\mathbf{y} = \mathbf{y}' \quad , \qquad \mathbf{R}_{cL\to cS}\cdot\mathbf{z} = \mathbf{z}'$$

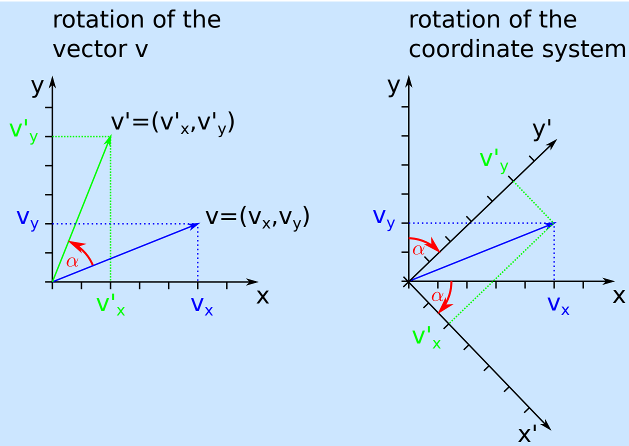
We already know the axes of both systems and find the rotation matrix as follows,

$$\mathbf{R}_{cL\to cS}\cdot\mathbf{x} = \begin{pmatrix}r_{c,xx} & r_{c,xy} & r_{c,xz}\\ r_{c,yx} & r_{c,yy} & r_{c,yz}\\ r_{c,zx} & r_{c,zy} & r_{c,zz}\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}r_{c,xx}\\ r_{c,yx}\\ r_{c,zx}\end{pmatrix} \overset{!}{=} \mathbf{x}' = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

By analogous computations for $\mathbf{y}$ and $\mathbf{z}$ we find

$$\mathbf{R}_{cL\to cS} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

However this is the rotation matrix for the coordinate systems. To find the rotation matrix for a vector remember the following.



As shown in the above sketch it does not matter whether you rotate the vector or the coordinate system, you will find the same coefficients for v'_x and v'_y. However, the rotation indicated by the red arrow is in the opposite direction. Thus the rotation for a vector is the inverse of the rotation for the coordinate system. Using also the unitarity $(R^{-1} = R^T)$ of rotation matrices we find

$$\mathbf{R}_{L\to S} = \mathbf{R}^{-1}_{cL\to cS} = \mathbf{R}^{T}_{cL\to cS} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

(b)
Recall that

$$\begin{aligned} \det(\mathbf{R}) &= \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix} \\ &= (R_{11}R_{22}R_{33} + R_{12}R_{23}R_{31} + R_{13}R_{21}R_{32}) - (R_{31}R_{22}R_{13} + R_{32}R_{23}R_{11} + R_{33}R_{21}R_{12})\,. \end{aligned}$$

For $\mathbf{R}_{L\to S}$ you thus find $\det(\mathbf{R}_{L\to S}) = 1$.

(c)
To keep it shorter we use here $\mathbf{R}$ for the rotation $\mathbf{R}_{L\to S}$. In dyadic notation the rotation of the second order tensor σ' can be found as explained in the lecture. We want to rotate the stress tensor from the specimen frame into the laboratory frame. By analysing the rotation behaviour of vectors we can figure out how one has to rotate a second order tensor. From (a) we know

$$\begin{aligned} & \mathbf{R}\cdot\mathbf{v} = \mathbf{v}' \\ \Leftrightarrow\quad & \mathbf{R}^T\cdot\mathbf{R}\cdot\mathbf{v} = \mathbf{R}^T\cdot\mathbf{v}' \\ \Leftrightarrow\quad & \mathbf{v} = \mathbf{R}^T\mathbf{v}' \end{aligned}$$

For a second order tensor $\mathbf{A}'$ and a vectors $\mathbf{v}'$ and $\mathbf{b}'$ in the specimen frame we find

$$\begin{aligned} & \mathbf{A}'\cdot\mathbf{v}' = \mathbf{b}' \\ \Leftrightarrow\quad & \mathbf{R}^T\cdot(\mathbf{A}'\cdot\mathbf{v}') = \mathbf{R}^T\cdot\mathbf{b}' \\ \Leftrightarrow\quad & \mathbf{R}^T\cdot(\mathbf{A}'\cdot\underbrace{\mathbf{R}\cdot\mathbf{R}^T}_{=\mathbb{1}}\cdot\mathbf{v}') = \underbrace{\mathbf{R}^T\cdot\mathbf{b}'}_{=\mathbf{b}} \\ \Leftrightarrow\quad & \underbrace{\mathbf{R}^T\cdot\mathbf{A}'\cdot\mathbf{R}}_{=\mathbf{A}}\cdot\underbrace{\mathbf{R}^T\cdot\mathbf{v}'}_{=\mathbf{v}} = \mathbf{b} \end{aligned}$$

So we have found:

$$\sigma = \mathbf{R}^T \cdot \sigma' \cdot \mathbf{R}$$

In Einstein sum notation one can write:

$$\sigma_{mn} = R_{im}\sigma'_{ij}R_{jn}.$$

Note that only $\sigma_{33} = \sigma_{zz}$ is nonzero. Therefore, we need to consider only one term for each mn, namely

$$\sigma_{mn} = R_{3m}R_{3n}\sigma'_{33} = R_{3m}R_{3n}\sigma_{zz}.$$

Furthermore, keep in mind that the rotated stress tensor must be symmetric. Hence

$$\begin{aligned}
\sigma_{11} &= R_{31}^2\sigma_{zz} = \frac{1}{3}\sigma_{zz}, \\
\sigma_{22} &= R_{32}^2\sigma_{zz} = \frac{1}{3}\sigma_{zz}, \\
\sigma_{33} &= R_{33}^2\sigma_{zz} = \frac{1}{3}\sigma_{zz}, \\
\sigma_{13} &= \sigma_{31} = R_{31}R_{33}\sigma_{zz} = \frac{1}{3}\sigma_{zz}, \\
\sigma_{23} &= \sigma_{32} = R_{32}R_{33}\sigma_{zz} = \frac{1}{3}\sigma_{zz}, \\
\sigma_{12} &= \sigma_{12} = R_{31}R_{32}\sigma_{zz} = \frac{1}{3}\sigma_{zz}.
\end{aligned}$$

An in matrix form

$$\boldsymbol{\sigma} = \frac{1}{3}\sigma_{zz}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(d)
Recall the definition of the von Mises stress in terms of the components of the stress tensor σ_{ij}, or in terms of the deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$$\sigma_{vM} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)}.$$

For $\boldsymbol{\sigma}'$ we have

$$\sigma'_{vM} = \sqrt{\frac{1}{2}\left[\sigma_{33}^2 + \sigma_{33}^2\right]} = \sigma_{zz}.$$

And for $\boldsymbol{\sigma}$

$$\sigma_{vM} = \sqrt{3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)} = \sigma_{zz}.$$

As expected, the two take the same value. The von Mises stress is an invariant of the stress tensor.

Question 2 . . . .

Analyse the plane stress

$$\underline{\sigma} = \begin{pmatrix} 3 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 2 \end{pmatrix}$$

(a) Find the rotation angle $\phi_{\sigma,\max}$ at which the diagonal stress entries are maximal.

(b) Which values take the principal or main stresses σ_1 and σ_2?

(c) Find the rotation angle $\phi_{\tau,\max}$ at which the shear stress is maximal and compute the value for the maximal shear stress $\tau_{\max}$.

(d) In the lecture it was shown that not only the principal stresses can characterize a stress state but also the stress invariants. Compute the stress invariants I_1 and I_2.

(e) The dimension of a stress as well as the dimension of the principal stresses is force per area. What are the dimensions of the two stress invariants I_1 and I_2.

Solution:

(a) In the lecture we have derived the formula

$$\tan(2\phi_{\sigma,\max}) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

By plugging in the numbers and equating out we find

$$\phi_{\sigma,\max} = \frac{\arctan(\sqrt{3})}{2} = \frac{\pi}{6} = 30°$$

(b) Method 1: use the formulas

$$\sigma_{1/2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{yy} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \frac{5}{2} \pm 1 = \begin{cases} \sigma_1 = \frac{7}{2} \\ \sigma_2 = \frac{3}{2} \end{cases}$$

Method 2: rotate the stress by the angle $\phi_{\sigma,\max} = 30°$

$$\underline{R} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \overset{\alpha=\phi_{\sigma,\max}}{=} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

(c) We use again the formula from the lecture

$$-\tan(2\phi_{\tau,\max}) = \frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}} \qquad \Rightarrow \qquad \phi_{\tau,\max} = -\frac{\arctan(\frac{1}{\sqrt{3}})}{2} = -\frac{\pi}{12} = -15° = \begin{cases} -15° \\ -15° + 90° \end{cases}$$

As in part (b) we can either use the formula

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm 1$$

or rotate the matrix by the angle $\phi_{\tau,\max} = \begin{cases} 15^\circ \\ 75^\circ \end{cases}$

$$\underline{R}_{-15^\circ} = \begin{pmatrix} \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} \\ -\frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} \end{pmatrix} \qquad \underline{R}_{75^\circ} = \begin{pmatrix} \frac{\sqrt{3}-1}{2\sqrt{2}} & -\frac{\sqrt{3}+1}{2\sqrt{2}} \\ \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix}$$

$$\sigma' = \underline{R}^T_{-15^\circ} \cdot \underline{\sigma} \cdot \underline{R}_{-15^\circ} = \begin{pmatrix} \frac{5}{2} & 1 \\ 1 & \frac{5}{2} \end{pmatrix} \qquad \sigma' = \underline{R}^T_{75^\circ} \cdot \underline{\sigma} \cdot \underline{R}_{75^\circ} = \begin{pmatrix} \frac{5}{2} & -1 \\ -1 & \frac{5}{2} \end{pmatrix}$$

(d) Method 1: The stress invariants I_i are the *negative* coefficients of the characteristic polynomial

$$\det(\sigma - \lambda \mathbb{1}) = \lambda^2 - 5\lambda + \frac{21}{4} \qquad \Rightarrow \qquad I_1 = 5\,, \quad I_2 = -\frac{21}{4}$$

Method 2: By the formulas derived in the lecture

$$I_1 = \mathrm{tr}(\sigma) = 5$$
$$I_2 = -\sigma_{xx}\sigma_{yy} + \tau_{xy}^2 = -\sigma_1\sigma_2 = -\det(\sigma) = -\frac{21}{4}$$

(e) I_1 is the trace of the stress and thus has the same dimension as the stress $(force/area)$. I_2 is the product of the two main stresses and thus hast the dimension $(force/area)^2$.

Question 3 . . . .

The following stress tensor characterises a special stress state

$$\underline{\sigma} = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

(a) Compute the angle $\phi_{\sigma,\max}$ at which the normal stresses takes its maximal value.

(b) Use the general rotation matrix and the computed angle $\phi_{\sigma,\max}$ to rotate the stress state in the coordinate system of maximal normal stress. What are the values for the principal stresses?

(c) What is the special name for the stress state found in (b)?

(d) Find the representation of the stress where the shear stress becomes maximal.

Solution:

(a) As in Q1 (a) we use the formula derived in the lecture and find

$$\phi_{\sigma,\max} = \frac{\arctan(1)}{2} = \frac{\pi}{8} = 22.5^\circ$$

(b) Now we are explicitly asked to rotate the stress into the coordinate system of maximal normal stresses.

$$\underline{R}_{22.5^\circ} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \overset{\alpha=\phi_{\sigma,\max}}{=} \begin{pmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & -\frac{\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{pmatrix}$$

$$\sigma_{\sigma,\max} = \underline{R}^T_{22.5^\circ} \cdot \sigma \cdot \underline{R}_{22.5^\circ} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

We find the two principal stresses $\sigma_1 = 2$, $\sigma_2 = -2$.

(c) The two principal stresses are the negative of each other. This stress state is called *pure shear* stress. By transforming into the coordinate system of maximal shear stress in part (d) we will see more clearly why this state is called pure shear.

(d) In the lecture we have shown that a rotation by $45°$ or $135°$ rotates a stress state from maximal normal stresses to maximal shear stresses. The two rotation matrices are given by

$$\underline{R}_{45°} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \underline{R}_{135°} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

You can use one of them to find the stress state in the coordinate system where the shear stress is maximal

$$\sigma_{\tau,\max} = \underline{R}_{45°}^T \cdot \underline{\sigma}_{\sigma,\max} \cdot R_{45°} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$
$$\sigma_{\tau,\max} = R_{135°}^T \cdot \sigma_{\sigma,\max} \cdot R_{135°} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Question 4 . . . .

Now we have a more general three dimensional stress state given by

$$\underline{\sigma} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

(a) Compute the three principal stresses which are the eigenvalues of the stress tensor.

(b) What are the values of the three invariants I_1, I_2 and I_3 of the given stress state?

(c) Compute the hydrostatic stress σ_h.

(d) Compute the deviatoric stress s_{ij} which is also called stress deviator.

(e) Which values take the invariants J_1, J_2 and J_3 of the stress deviator.

(f) What is the value of the von Mises stress?

(g) What is special about J_2 and why is the von Mises stress derived from J_2?

Solution:

(a) Compute the eigenvalues from the characteristic polynomial

$$\det(\sigma - \lambda \mathbb{1}) = -\lambda^3 + 6\lambda^2 + 8\lambda - 16 \overset{!}{=} 0$$

Either you use a computer or you can solve this equation also by hand. You can guess the solution $\lambda_1 = -2$ and use polynomial long division to find the other two roots.

$$(-\lambda^3 + 6\lambda^2 + 8\lambda - 16) : (\lambda + 2) = -\lambda^2 + 8\lambda - 8$$
$$-\lambda^2 + 8\lambda - 8 \overset{!}{=} 0 \quad \Rightarrow \quad \lambda_{2/3} = 2(2 \mp \sqrt{2})$$

So we find:

$$\sigma_1 = -2 \quad , \qquad \sigma_2 = 2(2 - \sqrt{2}) \quad , \qquad \sigma_3 = 2(2 + \sqrt{2})$$

(b) The three invariants I_1, I_2 and I_3 can be computed by the formulas given in the lecture or more easy can be directly read from the coefficients of the characteristic polynomial.

$$
\begin{aligned}
0 &= -\lambda^3 \underbrace{+6}_{=I_1} \lambda^2 \underbrace{+8}_{=I_2} \lambda \underbrace{-16}_{=I_3} \\
I_1 &= \mathrm{tr}(\sigma) = 1 + 4 + 1 = +6 \\
I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3 = -8 \\
I_3 &= \sigma_1\sigma_2\sigma_3 = -16
\end{aligned}
$$

(c) The hydrostatic stress in three dimensions is given by

$$\sigma_h = \frac{1}{3}\mathrm{tr}(\sigma) = \frac{1}{3} I_1 = +2$$

(d) The deviatoric stress s_{ij} is a pure shear stress and given by

$$s = \sigma - \sigma_h \mathbb{1} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & -1 \end{pmatrix}$$

(e) The invariants J_1, J_2 and J_3 of the stress deviator can be either be found by constructing the characteristic polynomial or by using the formulas from the lecture. As the trace of the deviatoric stress is zero by construction J_1 is always equal to zero.

$$
\begin{aligned}
0 &= -\lambda^3 \underbrace{+0}_{=J_1} \lambda^2 \underbrace{+20}_{=J_2} \lambda \underbrace{+16}_{J_3} \\
J_1 &= 0 \\
J_2 &= \frac{1}{6}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right) = +20 \\
J_3 &= \det(s) = +16
\end{aligned}
$$

(f) The von Mises stress can be computed directly from J_2

$$\sigma_{1/2}\mathrm{vM} = \sqrt{3J_2} = \sqrt{60} = 2\sqrt{15}$$

(g) The von Mises stress is a measure for a material if it undergoes plastic deformation or other failure. Typically materials only deform plastically under shear stress. In the lecture we have derived in two dimensions that $\sqrt{J_2}$ is equal to the maximal shear stress $\sigma_{\tau,\mathrm{max}}$. Thus the von Mises stress is the maximal shear stress and therefore a direct measure to propose if a material undergoes plastic deformation under a certain stress state.