Exercise 11: Rotation and invariants

Jan. 27-31

Question 1 . . . .

We want to demonstrate for the two-dimensional case that Hooke's law with isotropic elastic constants is indeed isotropic. Consider a 2D stress tensor σ and the corresponding strain ε,

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix}. \tag{1}$$

Next, consider the matrix for rotation by an arbitrary angle α

$$R = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}. \tag{2}$$

The most straightforward way to demonstrate isotropy would be to rotate the elastic stiffness tensor. However, this is a fourth-order tensor and rotating it is cumbersome. Here, we take a different approach. In order to demonstrate isotropy

1. express σ in terms of the components of ε,
2. rotate σ to find the representation σ' of this state of stress in the new coordinate system,
3. replace the components of ε in σ' by the components of the strain tensor ε' in the rotated coordinate system.

You should see that the constants of proportionality between stress and strain — the elastic constants — are the same in the new and the old coordinate system!

Solution:

1.)

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \tag{3}$$

$$\underline{\sigma} = \begin{bmatrix} 2\mu\varepsilon_{xx} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) & 2\mu\varepsilon_{xy} \\ 2\mu\varepsilon_{xy} & 2\mu\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) \end{bmatrix} \tag{4}$$

2.) Assuming that R is the matrix which, given the representation of a vector in the original coordinate system, yields the representation in the new coordinate system, we need to perform the following operation to find σ':

$$\begin{aligned} \sigma' &= R\sigma R^T \quad \text{(matrix notation), or, equivalently,} \\ \sigma'_{mn} &= R_{mi} R_{nj} \sigma_{ij} \quad \text{(index notation).} \end{aligned} \tag{5}$$

The result is

$$\begin{aligned} \sigma' &= \begin{bmatrix} \cos(\alpha)^2\sigma_{xx} + \sin(2\alpha)\sigma_{xy} + \sin(\alpha)^2\sigma_{yy} & \cos(2\alpha)\sigma_{xy} - \cos(\alpha)\sin(\alpha)(\sigma_{xx} - \sigma_{yy}) \\ \cos(2\alpha)\sigma_{xy} - \cos(\alpha)\sin(\alpha)(\sigma_{xx} - \sigma_{yy}) & \sin(\alpha)^2\sigma_{xx} - 2\sin(\alpha)\cos(\alpha)\sigma_{xy} + \cos(\alpha)^2\sigma_{yy} \end{bmatrix} \\ &= \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}, \quad \text{with} \\ \sigma'_{xx} &= (\varepsilon_{xx} + \varepsilon_{yy})(\mu + \lambda) + (\varepsilon_{xx} - \varepsilon_{yy})\mu\cos(2\alpha) + 2\varepsilon_{xy}\mu\sin(2\alpha), \\ \sigma'_{yy} &= (\varepsilon_{xx} + \varepsilon_{yy})(\mu + \lambda) - (\varepsilon_{xx} - \varepsilon_{yy})\mu\cos(2\alpha) - 2\varepsilon_{xy}\mu\sin(2\alpha), \\ \sigma'_{xy} &= \mu(2\varepsilon_{xy}\cos(2\alpha) - (\varepsilon_{xx} - \varepsilon_{yy})\sin(2\alpha)). \end{aligned} \tag{6}$$

3.) To get the components of ε in terms of the components of ε', we need to consider the reverse sense of rotation, i.e. $\varepsilon = R^T \varepsilon' R$. The transformation rules are the same for stress and strain, therefore the result can be obtained immediately by replacing $\alpha \to -\alpha$, $\sigma_{xx} \to \varepsilon'_{xx}$, $\sigma_{xy} \to \varepsilon'_{xy}$, and $\sigma_{yy} \to \varepsilon'_{yy}$ in the matrix above,

$$\varepsilon = \begin{bmatrix} \cos(\alpha)^2\varepsilon'_{xx} - \sin(2\alpha)\varepsilon'_{xy} + \sin(\alpha)^2\varepsilon'_{yy} & \cos(2\alpha)\varepsilon'_{xy} + \cos(\alpha)\sin(\alpha)\left(\varepsilon'_{xx} - \varepsilon'_{yy}\right) \\ \cos(2\alpha)\varepsilon'_{xy} + \cos(\alpha)\sin(\alpha)\left(\varepsilon'_{xx} - \varepsilon'_{yy}\right) & \sin(\alpha)^2\varepsilon'_{xx} + 2\sin(\alpha)\cos(\alpha)\varepsilon'_{xy} + \cos(\alpha)^2\varepsilon'_{yy} \end{bmatrix}. \tag{7}$$

Inserting the components of ε in the equations for σ'_{xx}, σ'_{xy}, and σ'_{yy}, we obtain

$$\sigma' = \begin{bmatrix} 2\mu\varepsilon'_{xx} + \lambda\left(\varepsilon'_{xx} + \varepsilon'_{yy}\right) & 2\mu\varepsilon'_{xy} \\ 2\mu\varepsilon'_{xy} & 2\mu\varepsilon'_{yy} + \lambda\left(\varepsilon'_{xx} + \varepsilon'_{yy}\right) \end{bmatrix}. \tag{8}$$

We can see that the elastic constants are the same in the two coordinate systems.

Question 2 ....

Reference: Rösler, Harders, Bäker, Mechanisches Verhalten der Werkstoffe, 2nd ed, Teubner, p. 412

A component made from a polycrystalline Aluminum alloy has the yield strength of 200 MPa and is subjected to plane stress with

$$\sigma_{xx} = \sigma_{yy} = 155 \text{ MPa}, \quad \tau_{xy} = 55 \text{ MPa}. \tag{9}$$

(a) Write the deviatoric stress!

(b) Calculate the principal stresses!

(c) Evaluate both Tresca's and von Mises' criterion to determine whether the material will yield!

Solution:

(a)

$$\sigma' = \sigma - \frac{1}{3}\operatorname{Tr}\sigma\mathbf{1} = \begin{bmatrix} 155 \text{ MPa} & 55 \text{ MPa} & 0 \\ 55 \text{ MPa} & 155 \text{ MPa} & 0 \\ 0 & 0 & 0 \end{bmatrix} - 310/3 \text{ MPa}\mathbf{1} = \begin{bmatrix} 155/3 \text{ MPa} & 55 \text{ MPa} & 0 \\ 55 \text{ MPa} & 155/3 \text{ MPa} & 0 \\ 0 & 0 & -310/3 \text{ MPa} \end{bmatrix} \tag{10}$$

(b)

$$\sigma_1 = 210 \text{ MPa}, \ \sigma_2 = 100 \text{ MPa}, \ \sigma_3 = 0 \text{ MPa} \tag{11}$$

(c) The two criteria contradict in this case,

$$\sigma_{\text{Tresca}} = \sigma_I - \sigma_{III} = 210 \text{ MPa} > 200 \text{ MPa} \implies \text{the material will yield,} \tag{12}$$

$$\sigma_{\text{Mises}} = \sqrt{\frac{1}{2}\left[\left(\sigma_I - \sigma_{II}\right)^2 + \left(\sigma_{II} - \sigma_{III}\right)^2 + \left(\sigma_I - \sigma_{III}\right)^2\right]} = 181.93 \text{ MPa} \implies \text{the material will not yield.} \tag{13}$$

Both criteria are empirical, hence no decision can be made.

Question 3 ....

Reference: Cleland *Foundations of Nanomechanics*, Springer, p. 174.

(a) A solid is subjected to the stress given below. Find the three stress invariants and the three principal values of stress. Solve for the directions of the three principal axes.

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ Pa} \tag{14}$$

(b) A solid is subjected to the stress given below. Find the expression for the stress tensor if the coordinate axes are rotated by $60°$ counterclockwise about the z-axis.

$$\boldsymbol{\sigma} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ Pa} \tag{15}$$

(c) A solid is stressed according to the tensor below. Show that for this form, the stress tensor is invariant under rotations about the z-axis.

$$\boldsymbol{\sigma} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \tag{16}$$

Solution:

(a) The three invariants are

$$\begin{aligned} I_1 &= tr(\sigma) = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 5 \text{ Pa}, \\ I_2 &= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{xz}^2 = 6 \text{ Pa}, \\ I_3 &= \det(\sigma) = 0 \text{ Pa}. \end{aligned} \tag{17}$$

The principal stresses are the eigenvalues λ_i $(i = 1 \ldots 3)$ of $\boldsymbol{\sigma}$, i.e. the roots of the characteristic polynomial. The invariants reappear here as the coefficients of the polynomial,

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0. \tag{18}$$

The roots are $\lambda_1 = 3$ Pa, $\lambda_2 = 2$ Pa, $\lambda_3 = 0$ Pa. The eigenvectors $\boldsymbol{e}_i$ $(i = 1 \ldots 3)$ are obtained by solving for the components of $\mathbf{e}_i$ in the equation

$$\boldsymbol{\sigma}\mathbf{e}_i = \lambda_i \mathbf{e}_i. \tag{19}$$

For each i, we have three equations and three unknowns (the components of $\mathbf{e}_i$),

$$\begin{aligned} \lambda_1 : &\quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} e_{1x} \\ e_{1y} \\ e_{1z} \end{bmatrix} = 3 \begin{bmatrix} e_{1x} \\ e_{1y} \\ e_{1z} \end{bmatrix} \rightarrow \mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ \lambda_2 : &\quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} e_{2x} \\ e_{2y} \\ e_{2z} \end{bmatrix} = 2 \begin{bmatrix} e_{2x} \\ e_{2y} \\ e_{2z} \end{bmatrix} \rightarrow \mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ \lambda_3 : &\quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} e_{3x} \\ e_{3y} \\ e_{3z} \end{bmatrix} = 0 \begin{bmatrix} e_{3x} \\ e_{3y} \\ e_{3z} \end{bmatrix} \rightarrow \mathbf{e}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{20}$$

(b) The rotation matrix that transforms from rotated to original frame is ($\theta = 60°$)

$$\boldsymbol{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

If we right-multiply this matrix with a column vector whose components are given in the rotated frame, then we get the representation of this vector in the original coordinate system. The rotated stress tensor is computed as

$$\begin{aligned} \boldsymbol{\sigma}' &= \mathbf{R}^T \boldsymbol{\sigma} \mathbf{R} \quad \text{(matrix notation)}, \\ \leftrightarrow \sigma'_{ij} &= R_{ki} R_{lj} \sigma_{kl} \quad \text{(index notation)}. \end{aligned} \tag{22}$$

Carrying out the multiplication, we find

$$\boldsymbol{\sigma}' = \frac{1}{2}\begin{bmatrix} \left(1+\sqrt{3}\right) & -\left(1+\sqrt{3}\right) & 0 \\ -\left(1+\sqrt{3}\right) & \left(3-\sqrt{3}\right) & 0 \\ 0 & 0 & 8 \end{bmatrix}. \quad (23)$$

(c) The rotation matrix has the form

$$\boldsymbol{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (24)$$

Rotation of the stress tensor gives

$$\boldsymbol{\sigma}' = \mathbf{R}^T \boldsymbol{\sigma} \mathbf{R} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} = \boldsymbol{\sigma}, \quad \text{independent of } \theta. \quad (25)$$

Q.E.D.