Exercise 11: Rotation and invariants
Jan. 27-31

Question 1 ....

We want to demonstrate for the two-dimensional case that Hooke's law with isotropic elastic constants is indeed isotropic. Consider a 2D stress tensor σ and the corresponding strain ε,

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix}. \tag{1}$$

Next, consider the matrix for rotation by an arbitrary angle α

$$R = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}. \tag{2}$$

The most straightforward way to demonstrate isotropy would be to rotate the elastic stiffness tensor. However, this is a fourth-order tensor and rotating it is cumbersome. Here, we take a different approach. In order to demonstrate isotropy

1. express σ in terms of the components of ε,
2. rotate σ to find the representation σ' of this state of stress in the new coordinate system,
3. replace the components of ε in σ' by the components of the strain tensor ε' in the rotated coordinate system.

You should see that the constants of proportionality between stress and strain — the elastic constants — are the same in the new and the old coordinate system!

Question 2 ....

Reference: Rösler, Harders, Bäker, Mechanisches Verhalten der Werkstoffe, 2nd ed, Teubner, p. 412

A component made from a polycrystalline Aluminum alloy has the yield strength of 200 MPa and is subjected to plane stress with

$$\sigma_{xx} = \sigma_{yy} = 155 \text{ MPa}, \quad \tau_{xy} = 55 \text{ MPa}. \tag{9}$$

(a) Write the deviatoric stress!
(b) Calculate the principal stresses!
(c) Evaluate both Tresca's and von Mises' criterion to determine whether the material will yield!

Question 3 ....

Reference: Cleland *Foundations of Nanomechanics*, Springer, p. 174.

(a) A solid is subjected to the stress given below. Find the three stress invariants and the three principal values of stress. Solve for the directions of the three principal axes.

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ Pa} \tag{14}$$

(b) A solid is subjected to the stress given below. Find the expression for the stress tensor if the coordinate axes are rotated by 60° counterclockwise about the z-axis.

$$\boldsymbol{\sigma} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ Pa} \tag{15}$$

(c) A solid is stressed according to the tensor below. Show that for this form, the stress tensor is invariant under rotations about the z-axis.

$$\boldsymbol{\sigma} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \tag{16}$$