Boundary conditions and beam stresses

Learning Objectives

After completing this chapter, you should be able to:

- **Apply** surface traction $\vec{t} = \underline{\sigma} \cdot \hat{n}$ to formulate boundary conditions on loaded or free surfaces
- **Enforce** traction and displacement continuity at material interfaces
- **Derive** the stress distribution $\sigma_{xx} = Mz/I_y$ inside beams from equilibrium and boundary conditions

Boundary conditions and surface tractions

The equilibrium equation $\nabla \cdot \underline{\sigma} = \vec{f}$ tells us how stresses must be distributed *inside* a solid body. But a differential equation alone does not uniquely determine the solution—we also need to specify what happens at the *boundaries* of the body. Think of it this way: knowing that water flows downhill (the governing equation) doesn't tell you where the water will go unless you also know where the riverbanks are (the boundary conditions).

In elasticity, boundary conditions describe how a solid interacts with its environment at its surfaces. What forces are being applied? Is the surface free to move, or is it constrained? These boundary conditions are essential for solving any practical problem.

Surface traction

Every physical object has boundaries—the surfaces where the material ends and the surrounding environment begins. At these surfaces, external forces may act on the body. We need a way to describe these surface forces that is compatible with our stress tensor formulation.

The **surface traction** (or simply **traction**) is the force per unit area acting on a surface. If you imagine a small patch of surface area ΔA experiencing a force $\Delta \vec{F}$, the traction is:

$$\vec{t} = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A}$$

You have already encountered a form of traction in the previous chapters: The line load $q(x)$, that describe the force per unit length on a beam. Dividing the line load by the width of the beam yields the normal traction on the surface.

The stress tensor provides a direct way to compute the traction on any surface. Given a surface with outward unit normal $\hat{n}$ (we use hats to denote unit vectors, i.e., vectors of length one), the traction vector is

$$\vec{t} = \underline{\sigma} \cdot \hat{n}, \tag{1}$$

or in component notation

$$t_i = \sum_{j=1}^{3} \sigma_{ij} n_j.$$

This is a fundamental relationship: once we know the stress tensor at a point, we can compute the force per unit area on *any* surface passing through that point—we just need to know the surface orientation $\hat{n}$.

The traction vector generally has three components: one normal to the surface (the **normal traction**) and two tangential to the surface (the **shear tractions**). For example, consider a horizontal surface with its normal pointing in the z-direction ($\hat{n} = \hat{z}$). The traction components are:

$$t_x = \sigma_{xz}, \quad t_y = \sigma_{yz}, \quad t_z = \sigma_{zz}$$

Here t_z is the normal traction (it pushes or pulls perpendicular to the surface), while t_x and t_y are shear tractions (they act parallel to the surface).

> **Visualizing traction**
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> Imagine pressing your hand flat against a table. The force your hand exerts on the table, divided by the contact area, is the traction. If you push straight down, you apply mostly normal traction. If you try to slide your hand while pressing down, you also apply shear traction. The traction vector captures both the magnitude and direction of this distributed surface force.

Traction boundary conditions

At the surface of a body, we can prescribe what traction must act there. This is called a **traction boundary condition** (also known as a Neumann boundary condition in the mathematical literature). Mathematically:

$$\underline{\sigma} \cdot \hat{n} = \vec{t}_0 \quad \text{on surface } S \tag{2}$$

where $\vec{t}_0$ is the prescribed traction vector and $\hat{n}$ is the outward normal to the surface.

Several important special cases arise frequently in practice:

Free surface ($\vec{t}_0 = 0$): No external force acts on the surface. This is the condition at any surface exposed to air (atmospheric pressure is usually negligible compared to stresses in solids). For a free surface with normal in the z-direction:

$$\sigma_{xz} = 0, \quad \sigma_{yz} = 0, \quad \sigma_{zz} = 0 \quad \text{at the surface}$$

Pressure loading ($\vec{t}_0 = -p\,\hat{n}$): A uniform pressure p acts normal to the surface, pushing inward. The negative sign indicates that positive pressure creates compressive (inward) force. This occurs when a solid is submerged in a fluid or enclosed in a pressurized chamber. The line load of the previous chapters describes such pressure loading.

Applied force: When a concentrated force $\vec{F}$ acts on a small area A, the traction is approximately $\vec{t}_0 = \vec{F}/A$. In reality, forces are always distributed over some area—point forces are idealizations that lead to stress singularities.

> **i** Why traction conditions fix three stress components
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> A traction boundary condition Equation 2 provides three equations (one for each component of $\vec{t}_0$). For a surface with normal $\hat{n} = \hat{z}$, these equations directly specify σ_{xz}, σ_{yz}, and σ_{zz} at the surface. The other stress components (σ_{xx}, σ_{yy}, σ_{xy}) are not constrained by this boundary condition—they are determined by equilibrium in the interior and other boundary conditions elsewhere.

Continuity conditions at interfaces

When two different materials are bonded together, the interface between them requires special consideration. Two fundamental conditions must be satisfied:

Traction continuity: The traction must be the same on both sides of the interface:

$$\underline{\sigma}^{(1)} \cdot \hat{n} = \underline{\sigma}^{(2)} \cdot \hat{n} \tag{3}$$

where superscripts (1) and (2) denote the two materials and $\hat{n}$ is the interface normal. This is simply Newton's third law: if material 1 pushes on material 2 with some force per unit area, then material 2 must push back on material 1 with an equal and opposite force per unit area.

Displacement continuity (for bonded interfaces): If the materials are bonded (glued, welded, etc.), they must move together:

$$\vec{u}^{(1)} = \vec{u}^{(2)}$$

If this condition is violated, the materials would separate or interpenetrate—which is physically impossible for a bonded interface.

> **i** Physical interpretation of traction continuity
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> Consider cutting an elastic body along any internal surface. The internal forces that were acting across this surface must be equal and opposite on the two newly created surfaces. This is simply Newton's third law applied to the internal stress distribution. The traction $\vec{t} = \underline{\sigma} \cdot \hat{n}$ represents this internal force per unit area.

Stresses inside beams

We now apply the equations above to a first problem: The stress inside of a beam. Consider a rectangular beam subject to bending. In the following derivation, we will assume that the stress fields do not vary in the y-direction (which is along the beam thickness). This means that all derivatives with respect to y will vanish. This is called a *plane* problem, in this case we will be working in what is called a *plane stress* situation.

The surface normal of the beam is oriented in z-direction and $z = 0$ is in the middle of the beam. Bending will give rise to internal stresses inside of the beam. We will require that these stresses comply with the external shear force $Q(x)$ and bending moment $M(x)$ in the *weak* or integral sense,

$$Q(x) = \int_A \mathrm{d}y\mathrm{d}z\, \tau_{xz}(x, z) \tag{4}$$

$$M(x) = \int_A \mathrm{d}y\mathrm{d}z\, z\sigma_{xx}(x, z). \tag{5}$$

Because of the plane state, integration in y-direction will only yield a constant factor, the width t of the beam.

We now simply take the derivatives of Equation 4 and Equation 5 with respect to the beam axis x. Taking the derivative of $Q(x)$ yields

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = \int_A \mathrm{d}y\mathrm{d}z\, \frac{\mathrm{d}\tau_{xz}}{\mathrm{d}x} = -\int_A \mathrm{d}y\mathrm{d}z\, \frac{\mathrm{d}\sigma_{zz}}{\mathrm{d}z} = -t\left[\sigma_{zz}(x, h/2) - \sigma_{zz}(x, -h/2)\right].$$

We call the quantity $q(x) \equiv t\left[\sigma_{zz}(x, h/2) - \sigma_{zz}(x, -h/2)\right]$ the line load of the beam. Next, we take the derivative of $M(x)$,

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \int_A \mathrm{d}y\mathrm{d}z\, z\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}x} = -\int_A \mathrm{d}y\mathrm{d}z\, z\frac{\mathrm{d}\tau_{xz}}{\mathrm{d}z} = -t\left[z\tau_{xz}\right]_{-h/2}^{h/2} + \int_A \mathrm{d}y\mathrm{d}z\, \tau_{xz}.$$

The first term on the right hand side vanishes because the surfaces are traction free, $\tau_{xz} = 0$ at $z = h/2$ and $z = -h/2$. The second term we recognize as the shear force $Q(x)$. We have therefore derived the equations

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -q(x) \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}x} = Q(x),$$

which are the equation for the equilibrium of internal forces in beams.

Applying traction boundary conditions to the beam

We have discussed traction boundary conditions in the previous section. Here we apply these concepts to the beam. The top and bottom surfaces of the beam (at $z = \pm h/2$) have normals pointing in the z-direction. As discussed in Equation 2, the traction boundary condition fixes three stress components at each surface: σ_{zz}, σ_{xz}, and σ_{yz}.
For a beam loaded only by a distributed line load $q(x)$ on its top surface:

- The shear stresses vanish at both surfaces: $\tau_{xz} = 0$ at $z = \pm h/2$ (traction-free in the tangential direction)
- The normal stress σ_{zz} at the surfaces relates to the applied load

Since the stress tensor is a field that varies throughout the body, these surface values are boundary conditions that constrain the solution in the interior.

To arrive at a closed-form solution for the stresses inside the beam, we now *assume* that the stress is a linear function of the position z perpendicular to the beam axis,

$$\sigma_{xx}(x, z) = C(x)z, \tag{6}$$

with some as-of-yet unknown function for the coefficient $C(x)$. In what follows we show that this is a good assumption, i.e., we can fulfill force and moment equilibrium and the resulting strains fulfill the compatibility conditions.

It is straightforward to compute the bending moment,

$$M(x) = \int_A \mathrm{d}y\mathrm{d}z\, \sigma_{xx}(x, z)z = C(x) \int_A \mathrm{d}y\mathrm{d}z\, z^2 = C(x)I_y, \tag{7}$$

where A is the cross-section of the beam and

$$I_y = \int_A \mathrm{d}y\mathrm{d}z\, z^2$$

is the *axial moment of inertia*. This is a number that characterizes the shape of the beam. For a rectangular beam of height h and width t, $I_y = h^3t/12$. The moment of inertia is a geometric factor and depends on the shape of the cross-section of the beam.

We can rewrite Equation 6 as

$$\sigma_{xx}(x,z) = \frac{M(x)}{I_y} z. \tag{8}$$

This equation is knows as Navier's bending formula, or just flexure formula. Note that an additional axial force $N(x)$ will simple be an additive contribution to Equation 8, $\sigma_{xx}(x,z) = M(x)/I_y z + N(x)/A$.

We can now use the condition for static equilibrium to compute the full stress tensor $\underline{\sigma}$. From $\partial_x \sigma_{xx} + \partial_z \tau_{xz} = 0$ we obtain

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{z}{I_y}\frac{\mathrm{d}M}{\mathrm{d}x} = -\frac{z}{I_y} Q(x).$$

We can integrate this across the height h of the beam, keeping in mind that $\tau_{xz} = 0$ at a traction-free surface, to yield

$$\tau_{xz}(x,z) = \frac{Q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right). \tag{9}$$

This equation is known as the Jourawski's formula or shear stress formula. Next, we use $\partial_z \sigma_{zz} + \partial_x \tau_{xz} = 0$ to obtain

$$\frac{\partial \sigma_{zz}}{\partial z} = -\frac{1}{2I_y}\frac{\mathrm{d}Q}{\mathrm{d}x}\left(\frac{h^2}{4} - z^2\right) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right).$$

We need to integrate this equation again, but now $\sigma_{zz} \neq 0$ at the surface since the beam is subject to a line load $q(x)$. Rather, we need the condition that the loads on top and bottom surface of the beam balance, $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2)$. This gives

$$\sigma_{zz}(x,z) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) z \tag{10}$$

with

$$\sigma_{zz}(x, h/2) = \frac{q(x)h^3}{24I_y} = \frac{q(x)}{2t} \tag{11}$$

where t is the width of the beam.

> **ℹ Connection to Euler-Bernoulli beam theory**
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> In this section, we derived the stress tensor inside a beam using the plane stress approximation of Linear Elasticity. This is a generalization that includes the the *Euler-Bernoulli beam theory* we have seen in previous chapters. However, the starting point of this theory is typically not Equation 6, but the assumption that each cross section will remain plane and undergo small rotations during deformation. These are sometimes called the *Bernoulli*

assumptions and are valid if the beam is slender ($h << L$). The Euler-Bernoulli theory implies that the strain $\varepsilon_{xx} \propto z$ rather than the stress. It follows that $\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$, but this of course ignores the other component of the stress tensor σ_{zz}. In fact, according to plain stress conditions in Linear Elasticity $\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{zz}}{E}$. However, we have seen that $\sigma_{zz} \propto qh^2$, whereas $\sigma_{xx} \propto M \approx qL^2$. Thus, it is reasonable for slender beams ($h << L$) to consider $\sigma_{zz} \approx 0$, as the axial stress is predominant. It is also interesting to notice that σ_{zz} vanishes anyway if there is no distributed load. Therefore, the critical stresses in slender beams are always caused by the bending moment, making the Euler-Bernoulli theory a powerful and valid tool.

Chapter Summary

This chapter covered boundary conditions and their application to beam stresses:

- **Surface traction**: $\vec{t} = \underline{\sigma} \cdot \hat{n}$ is the force per unit area on a surface with normal $\hat{n}$
- **Traction boundary conditions**: Prescribe $\vec{t} = \vec{t}_0$ on surfaces; free surfaces have $\vec{t} = 0$
- **Interface conditions**: Traction continuity (Newton's third law) and displacement continuity (bonded interfaces)
- **Beam stress distribution**: $\sigma_{xx} = Mz/I_y$ varies linearly through the cross-section
- **Shear stress**: $\tau_{xz} = Q(h^2/4 - z^2)/(2I_y)$ is parabolic through the thickness
- **Moment of inertia**: $I_y = \int z^2 \, dA$ is a geometric property ($I_y = h^3t/12$ for rectangular beams)
- **Equilibrium relations**: $dQ/dx = -q$ and $dM/dx = Q$ follow from 3D equilibrium with traction BCs

Boundary conditions are essential for solving elasticity problems—they specify how a body interacts with its environment.