Hooke's law

Learning Objectives

After completing this chapter, you should be able to:

- **State** Hooke's law for isotropic materials using elastic constants (E, ν) or (λ, μ)
- **Apply** Hooke's law to calculate stress from strain (and vice versa) in 3D and plane problems
- **Distinguish** between plane stress and plane strain conditions and convert between them

Hooke's law

General form

In order to compute the deformation of an elastic body, we need a constitutive relation (material law) to close the equations of elastostatic equilibrium,

$$\nabla \cdot \underline{\sigma} = 0 \quad \text{and} \quad \underline{\varepsilon}(\vec{r}) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right). \tag{1}$$

Since we will be working in linear elasticity, the constitutive equation is a linear relationship between $\underline{\sigma}$ and $\underline{\varepsilon}$. The most general form of this linear relationship is

$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}, \text{or using Einstein summation } \sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{2}$$

It is called *Hooke's law*. The quantity $\underline{\underline{C}}$ is a fourth order symmetric tensor of elastic constants that contains at most 21 independent elastic moduli. To see that there are only 21 independent coefficients, it is useful to remove the symmetric entries from $\underline{\sigma}$ and $\underline{\varepsilon}$ and express them as 6-vectors in what is called * Voigt notation,

$$\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}) \tag{3}$$

and

$$\vec{\varepsilon} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{yz}, 2\varepsilon_{xz}, 2\varepsilon_{xy}). \tag{4}$$

Then $\sigma = \underline{C} \cdot \vec{\varepsilon}$ *where* $\underline{C}$ *is a* 6×6 *symmetric matrix called the* stiffness matrix * containing the above-mentioned 21 independent elastic constants. (There are $6 \cdot 6 = 36$ components, but the matrix is symmetric.)

Note that the off-diagonal components of $\underline{\sigma}$ are often denoted by $\tau_{xy} \equiv \sigma_{xy}, \tau_{xz} \equiv \sigma_{xz}$ and $\tau_{yz} \equiv \sigma_{yz}$. The off-diagonal components of $\underline{\varepsilon}$ are often denoted by $\gamma_{xy} \equiv 2\varepsilon_{xy}, \gamma_{xz} \equiv 2\varepsilon_{xz}$ and $\gamma_{yz} \equiv 2\varepsilon_{yz}$ and absorb the factor of 2 that occurs in Eq. (4). Voigt notation then becomes

$$\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{yz}, \tau_{xz}, \tau_{xy}) \tag{5}$$

and

$$\vec{\varepsilon} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}). \tag{6}$$

Note: It is important to keep in mind that the $ $'s contain a factor 2 but the $ $'s do not. The factor of 2 ensures that $\vec{\sigma} = \underline{C} \cdot \vec{\varepsilon}$ and $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$ are the same constitutive law.

Isotropic solids

For isotropic elasticity, the total 21 independent elastic constants reduce to two. The constitutive equation for isotropic elasticity is

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \tag{7}$$

or its inverse

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\delta_{kk} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}], \tag{8}$$

where δ_{ij} is the Kronecker-Delta. These expressions have been conveniently written in their most simple form. The constants that show up in Eqs. (7) and (8) are the shear modulus $ $, Lamé's first constant $ $, Young's modulus E and Poisson number $ $. Both $ $ and $ $ are often called Lamé's constants. Note that $\sigma_{kk} = 3\sigma_h$ (Einstein summation!) where σ_h is the hydrostatic stress.

The four moduli are not independent (only two are), and the following expressions relate the pairs $ $, $ $ and E, $ $ to each other:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{9}$$

$$\mu = \frac{E}{2(1+\nu)} \tag{10}$$

$$\lambda + \mu = \frac{E}{2(1+\nu)(1-2\nu)} \tag{11}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{12}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{13}$$

Note that another common symbol for the shear modulus $ $ is the Latin letter

G.

The volumetric strain $\varepsilon_h = \frac{1}{3}$ tr $\underline{\varepsilon} = \frac{1}{E}[(1+\nu)\sigma_h - 3\,\nu\sigma_h] = \frac{1}{E}(1-2\nu)\sigma_h$ vanishes at $\nu = 1/2$. In this case, $\sigma_{ij} = 2\sigma_h\varepsilon_{ij}$ because the $\varepsilon_h = \varepsilon_{kk}$ must vanish.

We can also write down a free energy functional (often also called a hyperelastic energy density), which is quadratic in the strain $\underline{\varepsilon}$,

$$W = \frac{1}{2}\lambda\varepsilon_{ii}^2 + \mu\varepsilon_{ij}^2 \tag{14}$$

Using $\sigma_{ij} = \partial W/\partial\varepsilon_{ij}$ recovers the above constitutive expression Eq. (7). From the free energy functional we see that any isotropic material must have $\lambda > 0$ and $\mu > 0$, otherwise the energy could be made arbitrarily small by increasing the deformation of the solid. This limits the Poisson number to the range $ -1 < < 1/2 $. Note that $\nu < 0$ is typically only achieved for architectured materials such as foams or metamaterials.

Example: Uniaxial tension test

Tip

Consider a rectangular block of isotropic elastic material subjected to uniaxial tension. The block has Young's modulus $E = 200$ GPa and Poisson's ratio $\nu = 0.3$. An axial stress $\sigma_{xx} = 100$ MPa is applied in the x-direction with all other stresses zero: $\sigma_{yy} = \sigma_{zz} = \tau_{yz} = \tau_{xz} = \tau_{xy} = 0$.
Find:

- The axial strain ε_{xx}
- The transverse strains ε_{yy} and ε_{zz}
- The volumetric strain ε_h

Solution:
This is the simplest and most fundamental loading case in elasticity: a bar being stretched in one direction. We want to find how much the bar elongates in the loading direction and how much it contracts in the perpendicular directions.
We use the inverse constitutive relation (Eq. (8)) that gives strain in terms of stress:

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$$

This formula contains the trace of the stress tensor, σ_{kk}, which is the sum of the diagonal components. We calculate this first:

$$\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 100 + 0 + 0 = 100 \text{ MPa}$$

Now we apply the formula to find each strain component. For the axial strain ε_{xx}, we set $i = j = x$. The Kronecker delta $\delta_{xx} = 1$, so:

$$\varepsilon_{xx} = \frac{1}{E}[(1+\nu)\sigma_{xx} - \nu\sigma_{kk}]$$

Substituting the values ($E = 200$ GPa $= 200 \times 10^3$ MPa, $\nu = 0.3$, $\sigma_{xx} = 100$ MPa, $\sigma_{kk} = 100$ MPa):

$$\varepsilon_{xx} = \frac{1}{200 \times 10^3}[(1+0.3)(100) - (0.3)(100)] = \frac{1}{200 \times 10^3}[130 - 30] = \frac{100}{200 \times 10^3}$$

$$= 0.5 \times 10^{-3} = 5 \times 10^{-4}$$

This means the bar elongates by 0.5 mm for every meter of original length (or equivalently, 0.05%).

For the transverse strain ε_{yy}, we set $i = j = y$. Again $\delta_{yy} = 1$, and now $\sigma_{yy} = 0$:

$$\varepsilon_{yy} = \frac{1}{E}[(1+\nu)\sigma_{yy} - \nu\sigma_{kk}] = \frac{1}{200 \times 10^3}[(1.3)(0) - (0.3)(100)]$$

$$= \frac{1}{200 \times 10^3}[-30] = -0.15 \times 10^{-3} = -1.5 \times 10^{-4}$$

The negative sign indicates contraction: the bar gets thinner as it stretches. By symmetry of the loading and material (isotropic), the z-direction behaves the same as the y-direction: $\varepsilon_{zz} = -0.15 \times 10^{-3}$.

The volumetric strain ε_h (also called dilatation) is the sum of the normal strains:

$$\varepsilon_h = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = (0.5 - 0.15 - 0.15) \times 10^{-3} = 0.2 \times 10^{-3}$$

We can verify this using the alternative formula from the text. The hydrostatic stress is $\sigma_h = \sigma_{kk}/3 = 100/3$ MPa, and:

$$\varepsilon_h = \frac{1}{E}(1-2\nu)\sigma_{kk} = \frac{1}{200 \times 10^3}(1-0.6)(100) = \frac{40}{200 \times 10^3} = 0.2 \times 10^{-3}$$

This confirms our calculation. The positive volumetric strain means the material increases in volume when stretched—the elongation in the x-direction is not fully compensated by the contractions in the y and z directions. This is true for all materials with $\nu < 0.5$. An incompressible material ($\nu = 0.5$) would have $\varepsilon_h = 0$.

Finally, we can verify that the Poisson's ratio is recovered correctly. The ratio of transverse to axial strain should equal $-\nu$:

$$\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{-0.15 \times 10^{-3}}{0.5 \times 10^{-3}} = -0.3 = -\nu \quad \checkmark$$

Plane strain

Just as plane stress applies to thin structures, **plane strain** applies to thick or constrained structures where deformation in one direction is prevented.

When does plane strain apply?

In plane strain, the strain in one direction (say the y-direction) is constrained to be zero:

$$\varepsilon_{yy} = 0, \quad \gamma_{xy} = 0, \quad \gamma_{yz} = 0$$

This condition applies when:

- The structure is very long in the y-direction compared to x and z
- The ends in the y-direction are constrained (e.g., between rigid walls)
- The loading and geometry do not vary along the y-direction

Common examples include:

- Long tunnels or pipes under pressure
- Dams and retaining walls
- Any structure where one dimension is much larger than the others

Hooke's law for plane strain

Under the constraint $\varepsilon_{yy} = 0$, the stress σ_{yy} is generally non-zero (stress develops to enforce the constraint). From Hooke's law:

$$\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] = 0$$

This gives:

$$\sigma_{yy} = \nu(\sigma_{xx} + \sigma_{zz})$$

Substituting back into Hooke's law for the remaining strains:

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{zz} \tag{15}$$

$$\varepsilon_{zz} = \frac{1-\nu^2}{E}\sigma_{zz} - \frac{\nu(1+\nu)}{E}\sigma_{xx} \tag{16}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} \tag{17}$$

The inverse relations (stress from strain) are:

$$\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{zz} \tag{18}$$

$$\sigma_{zz} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{zz} \tag{19}$$

$$\tau_{xz} = G\gamma_{xz} \tag{20}$$

Comparison: Plane stress vs. plane strain

Both conditions reduce 3D elasticity to 2D, but they apply to different physical situations:

Property	Plane Stress	Plane Strain
Zero component	$\sigma_{yy} = 0$	$\varepsilon_{yy} = 0$
Structure type	Thin (small in y)	Thick/long (large in y)
Consequence	$\varepsilon_{yy} \neq 0$ (free contraction)	$\sigma_{yy} \neq 0$ (constraint stress)
Examples	Thin plates, beams	Dams, tunnels, retaining walls

Converting between plane stress and plane strain

The equations for both conditions have the same mathematical form. A plane stress solution can be converted to plane strain by substituting:

$$E \to \frac{E}{1-\nu^2}, \quad \nu \to \frac{\nu}{1-\nu}$$

This allows solutions derived for one condition to be readily adapted to the other, which is extremely useful in practice.

Chapter Summary

This chapter presented Hooke's law for linear elastic materials:

- **General form**: $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$ relates stress and strain via the stiffness tensor
- **Isotropic materials**: Reduce to 2 independent constants; commonly E, ν or λ, μ
- **Voigt notation**: Represents stress/strain as 6-vectors; stiffness as 6×6 matrix
- **Forward relation**: $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$ gives stress from strain
- **Inverse relation**: $\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$ gives strain from stress
- **Conversion formulas**: $\mu = E/[2(1+\nu)]$, $\lambda = E\nu/[(1+\nu)(1-2\nu)]$
- **Incompressibility limit**: At $\nu = 1/2$, volumetric strain vanishes
- **Plane strain**: $\varepsilon_{yy} = 0$ for thick/constrained structures; $\sigma_{yy} = \nu(\sigma_{xx} + \sigma_{zz})$
- **Conversion**: Plane stress   plane strain via $E \to E/(1-\nu^2)$, $\nu \to \nu/(1-\nu)$

Hooke's law is the constitutive relation that closes the elasticity equations and enables calculation of deformations.