Plane problems and compatibility

Learning Objectives

After completing this chapter, you should be able to:

- **Derive** the compatibility condition $\varepsilon_{xx,zz}+\varepsilon_{zz,xx}=2\varepsilon_{xz,xz}$ from strain-displacement relations
- **Explain** why compatibility ensures strain fields arise from continuous displacement fields
- **Apply** compatibility to verify whether a given strain field is physically realizable

Plane problems

Plane problems are problems where the system has symmetry in a certain direction. We use the y-direction as the direction in which the plane conditions hold, meaning that all quantities are independent of y.

As discussed in earlier chapters, there are two main types of plane problems:

- **Plane strain** ($\varepsilon_{yy}=0$): Covered in the chapter on Hooke's law; applies to thick or constrained structures
- **Plane stress** ($\sigma_{yy}=0$): Covered in the chapter on beam stresses; applies to thin structures

Both conditions reduce full 3D elasticity to 2D analysis in the x-z plane. The equilibrium condition in 2D becomes:

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xz}}{\partial z}=0 \tag{1}$$

$$\frac{\partial \sigma_{zz}}{\partial z}+\frac{\partial \sigma_{xz}}{\partial x}=0 \tag{2}$$

These two equations, together with Hooke's law (in plane strain or plane stress form) and the compatibility condition discussed below, form the complete set of equations for plane elasticity problems.

Compatibility condition

The **compatibility condition** is a constraint that every valid strain field must satisfy. It ensures that the strain field can actually arise from a continuous displacement field.

Why compatibility is needed

In elasticity, we often work with stress and strain fields rather than displacements. When we calculate strains from stresses using Hooke's law, we need to ensure that these strains are *compatible*—that is, they can be integrated to give a single-valued, continuous displacement field.

Not every arbitrary strain field can arise from an actual deformation. The strain components must satisfy certain differential relationships to ensure geometric consistency.

Derivation

For plane problems, the strain-displacement relations are:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{3}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{4}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \tag{5}$$

We can eliminate the displacement components by taking appropriate derivatives. From the first equation:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^3 u_x}{\partial x \partial z^2}$$

From the second equation:

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = \frac{\partial^3 u_z}{\partial z \partial x^2}$$

From the third equation:

$$2\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = \frac{\partial^3 u_x}{\partial x \partial z^2} + \frac{\partial^3 u_z}{\partial z \partial x^2}$$

Combining these, we obtain the **compatibility condition** for plane problems:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} \tag{6}$$

This is a necessary and sufficient condition (in a simply-connected domain) for a strain field to be derivable from a single-valued displacement field.

Geometric interpretation: The jigsaw puzzle analogy

The compatibility condition has a simple geometric explanation. Imagine a jigsaw puzzle that you deform in its assembled state. Even in the deformed state, all pieces must still fit together—there can be no gaps or overlaps.

If you were to deform each puzzle piece independently (ignoring the constraint that they must fit together), you could assign arbitrary strains to each piece. But when you try to reassemble the puzzle, the pieces might not fit.

The compatibility condition ensures that neighboring material elements deform in a way that they remain geometrically compatible—they continue to fit together after deformation. This is mathematically equivalent to requiring that the displacement field be single-valued and continuous.

> **ℹ Connection to vector calculus**
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> The compatibility condition is analogous to a well-known result in vector calculus: the curl of a gradient is zero ($\nabla \times (\nabla\phi) = 0$).
> Just as this condition ensures that a vector field is the gradient of some scalar potential, the compatibility conditions ensure that a strain tensor field is the symmetrized gradient of some displacement vector field.

Compatibility in 3D

In three dimensions, there are six compatibility equations (one for each pair of strain components). The 2D compatibility equation derived above is a special case that applies when quantities are independent of the y-direction.

Practical importance

In solving elasticity problems:

1. **Displacement-based methods**: If we solve for displacements directly, compatibility is automatically satisfied (since we compute strains as derivatives of displacements)
2. **Stress-based methods**: If we work with stress fields (e.g., using the Airy stress function), we must explicitly verify that the resulting strains satisfy compatibility

The compatibility condition plays a central role in the mathematical structure of elasticity theory and is essential for methods like the Airy stress function approach used in fracture mechanics.

Chapter Summary

This chapter covered the compatibility condition for plane problems:

- **Plane problems**: Reduce 3D elasticity to 2D analysis (plane strain or plane stress)
- **Equilibrium in 2D**: $\partial_x \sigma_{xx} + \partial_z \sigma_{xz} = 0$ and $\partial_z \sigma_{zz} + \partial_x \sigma_{xz} = 0$
- **Compatibility condition**: $\varepsilon_{xx,zz} + \varepsilon_{zz,xx} = 2\varepsilon_{xz,xz}$
- **Physical meaning**: Ensures strain fields arise from continuous displacement fields
- **Jigsaw analogy**: Deformed pieces must still fit together—no gaps or overlaps
- **Vector calculus connection**: Analogous to "curl of gradient equals zero"

Compatibility ensures that solutions are physically realizable and is essential for stress-based solution methods.