Back to beams

Learning Objectives

After completing this chapter, you should be able to:

- **Apply** plane stress conditions to derive the stress distribution $\sigma_{xx} = Mz/I_y$ in bent beams
- **Derive** the Euler-Bernoulli beam equation from 3D elasticity via Hooke's law
- **Calculate** critical buckling loads using the Euler formula $N_{\text{cr}} = \pi^2 EI/(KL)^2$

Plane stress conditions

Before analyzing stresses in beams, we introduce the concept of **plane stress**, which greatly simplifies the 3D elasticity equations for thin structures like beams and plates.

When does plane stress apply?

In full 3D elasticity, the stress tensor $\underline{\sigma}$ has six independent components. However, for structures that are thin in one direction (say, the y-direction), we can often assume that the stress components involving this direction vanish:

$$\sigma_{yy} = 0, \quad \tau_{xy} = 0, \quad \tau_{yz} = 0$$

This is called **plane stress** and is an excellent approximation when:

- The structure is thin (small extent in the y-direction compared to x and z)
- No forces are applied in the y-direction
- The surfaces perpendicular to y are traction-free

Under these conditions, the only non-zero stress components are:

$$\sigma_{xx}, \quad \sigma_{zz}, \quad \tau_{xz}$$

These stresses vary only in the x-z plane, hence the name "plane stress."

Simplified Hooke's law for plane stress

Under plane stress conditions, Hooke's law simplifies considerably. For an isotropic material with $\sigma_{yy} = 0$:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{zz}) \tag{1}$$
$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx}) \tag{2}$$
$$\varepsilon_{yy} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) \tag{3}$$
$$\gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{2(1+\nu)}{E}\tau_{xz} \tag{4}$$

Note that even though $\sigma_{yy} = 0$, the strain ε_{yy} is generally non-zero (the material contracts or expands freely in the y-direction due to the Poisson effect).

The inverse relations give stress from strain:

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{zz}) \tag{5}$$
$$\sigma_{zz} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{zz}) \tag{6}$$
$$\tau_{xz} = G\gamma_{xz} \tag{7}$$

Application to beams

For a beam oriented along the x-axis with thickness in the z-direction and width in the y-direction, plane stress conditions apply when:

- The beam width (in y) is small compared to other dimensions
- The beam surfaces at $y = \pm t/2$ (where t is the width) are stress-free

This allows us to treat beam bending as a 2D problem in the x-z plane, which we now proceed to analyze.

Beams

Stresses

Consider a rectangular beam in plane stress subject to bending. (The "plane" direction is the y-direction.) The surface normal of the beam is oriented in z-direction and $z = 0$ is in the middle of the beam. Bending will give rise to internal stresses inside of the beam. We will require that these stresses comply with the external shear force $Q(x)$ and bending moment $M(x)$ in the *weak* or integral sense,

$$Q(x) = \int_A \mathrm{d}y\mathrm{d}z\, \tau_{xz}(x, z) \tag{8}$$

$$M(x) = \int_A \mathrm{d}y\mathrm{d}z\, z\sigma_{xx}(x, z). \tag{9}$$

Because of the plane state, integration in y-direction will only yield a constant factor, the width t of the beam.

We can derive a condition equivalent to static equilibrium, $\sigma_{ij,j} = 0$, for the weak quantities defined in Eqs. (8) and (9). Taking the derivative of $Q(x)$ yields

$$Q_{,x} = \int_A \mathrm{d}y\mathrm{d}z\, \tau_{xz,x} = -\int_A \mathrm{d}y\mathrm{d}z\, \sigma_{zz,z} = -t\left[\sigma_{zz}(x, h/2) - \sigma_{zz}(x, -h/2)\right]. \tag{10}$$

We call the quantity $q(x) \equiv t\left[\sigma_{zz}(x, h/2) - \sigma_{zz}(x, -h/2)\right]$ the line load of the beam. Next, we take the derivative of $M(x)$,

$$M_{,x} = \int_A \mathrm{d}y\mathrm{d}z\, z\sigma_{xx,x} = -\int_A \mathrm{d}y\mathrm{d}z\, z\tau_{xz,z} = -t\left[z\tau_{xz}\right]_{-h/2}^{h/2} + \int_A \mathrm{d}y\mathrm{d}z\, \tau_{xz}. \tag{11}$$

The first term on the right hand side vanishes because the surfaces are traction free, $\tau_{xz} = 0$ at $z = h/2$ and $z = -h/2$. This yields

$$Q_{,x} = -q(x) \tag{12}$$

$$M_{,x} = Q(x) \tag{13}$$

for the weak form of the equilibrium conditions.

Note: We have not yet talked about free surfaces, but of course every physical object will be bounded by surfaces. The surface can be loaded, i.e. there can a force acting on it. This force, normalized by area, is called a * tractions. *(We have already encountered a type of traction in form of the* line load * in the first chapters.) The traction is a vector field (defined over the surface of our object) with components t_x, t_y and t_z. Note that in the literature these are often denoted by P for the normal component, and Q_x, Q_y for the components perpendicular to the surface. (Attention: Do not confuse this with the Q that we call the shear force here!)

These traction enter our equations as boundary condition. They fix three components of the stress tensor at the surface. For a surface with a normal in z-direction,

$$\sigma_{zz} = -t_z, \quad \sigma_{xz} = t_x \quad \text{and} \quad \sigma_{yz} = t_y. \tag{14}$$

For a traction-free surface,

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \tag{15}$$

at the surface. Since the stress-tensor is a tensor field that varies throughout the body, these components will of course deviate from the surface tractions when we regard points in the interior of our (deformed) body. We now * assume * that the stress is a linear function of the position z perpendicular to the beam axis,

$$\sigma_{xx}(x, z) = C(x)z. \tag{16}$$

In what follow we show that this is a good assumption, i.e. we can fulfill force and moment equilibrium and the resulting strains fulfill the compatibility conditions.

Note: The theory derived in this chapter is commonly referred to as the * Euler-Bernoulli beam theory*. The starting point of this theory is typically not Eq. (16), but the assumption that each cross section will remain plane and undergo small rotations during deformation. These are sometimes called the* Bernoulli assumptions*. They imply that the strain $\varepsilon_{xx} \propto z$ rather than the stress. It is often argued that $\sigma_{xx} = E\varepsilon_{xx}$ but this of course ignores the other components of the strain tensor. As will be seen below, the Bernoulli assumptions are actually wrong but assuming a linear stress profile leads to the correct small strain expression for the deformation of a beam.

It is straightforward to compute the bending moment,

$$M(x) = \int_A \mathrm{d}y\mathrm{d}z\, \sigma_{xx}(x, z)z = C(x) \int_A \mathrm{d}y\mathrm{d}z\, z^2 = C(x)I_y, \tag{17}$$

where A is the cross-section of the beam and

$$I_y = \int_A \mathrm{d}y\mathrm{d}z\, z^2 \tag{18}$$

is the * axial moment of inertia*. For a rectangular beam of height h and width t, $I_y = h^3t/12$. The moment of inertia is a geometric factor and depends on the shape of the cross-section of the beam.

We can rewrite Eq. (16) as

$$\sigma_{xx}(x, z) = \frac{M(x)}{I_y} z. \tag{19}$$

Note that an additional longitudinal force L will simple be an additive contribution to Eq. (19), $\sigma_{xx}(x,z) = M(x)/I_y z + L/A$.

We can now use the condition for static equilibrium to compute the full stress tensor $\underline{\sigma}$. From $\sigma_{xx,x} + \tau_{xz,z} = 0$ we obtain

$$\tau_{xz,z} = -\frac{z}{I_y} M_{,x} = -\frac{z}{I_y} Q(x). \tag{20}$$

We can integrate this across the height h of the beam, keeping in mind that $\tau_{xz} = 0$ at a traction-free surface, to yield

$$\tau_{xz}(x,z) = \frac{Q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right). \tag{21}$$

Next, we use $\sigma_{zz,z} + \tau_{xz,x} = 0$ to obtain

$$\sigma_{zz,z} = -\frac{1}{2I_y} Q_{,x}\left(\frac{h^2}{4} - z^2\right) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right). \tag{22}$$

We need to integrate this equation again, but now $\sigma_{zz} \neq 0$ at the surface since the beam is subject to a line load $q(x)$. Rather, we need the condition that the loads on top and bottom surface of the beam balance, $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2)$. This gives

$$\sigma_{zz}(x,z) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) z \tag{23}$$

with

$$\sigma_{zz}(x, h/2) = \frac{q(x) h^3}{24 I_y} = \frac{q(x)}{2t} \tag{24}$$

where t is the width of the beam.

Note that in this derivation, we have required that the stress gives rise to a certain bending moment through Eq. (17). Since we do not prescribe the specific stress state $\sigma_{xx}(x,z)$ but only its integral, this is a * weak * condition. Similarly, integration of Eq. (21) in accordance with Eq. (8) gives

$$\int_{-t/2}^{t/2} \mathrm{d}y \int_{-h/2}^{h/2} \mathrm{d}z\, \tau_{xz}(x,z) = Q(x), \tag{25}$$

i.e. the force acting on the cross-section at position x along the beam. Again, this condition is fulfilled in the integral, i.e. in the weak sense.

Displacements

Now that we know the stress inside the beam, we can compute the displacement and thereby the deformation of the beam from Hooke's law. Starting from Hooke's law (in plain stress, i.e. for $\sigma_{yy} = 0$),

$$\varepsilon_{xx} \equiv u_{x,x} = \sigma_{xx}/E - \nu\sigma_{zz}/E \tag{26}$$

$$2\varepsilon_{xz} \equiv u_{x,z} + u_{z,x} = 2(1+\nu)\tau_{xz}/E, \tag{27}$$

and taking the derivative of Eq. (27) with respect

to x, we obtain

$$u_{x,xz} + u_{z,xx} = 2(1+\nu)\tau_{xz,x}/E \tag{28}$$

and

$$u_{x,xz} + u_{z,xx} = \partial_z(u_{x,x}) + u_{z,xx} = u_{z,xx} + (\sigma_{xx,z} - \nu\sigma_{zz,z})/E. \tag{29}$$

Combining these two equations yields

$$u_{z,xx} = \left[2(1+\nu)\tau_{xz,x} - \sigma_{xx,z} + \nu\sigma_{zz,z}\right]/E, \tag{30}$$

where we now insert Eqs. (19), (21) and (23). This gives

$$u_{z,xx} = \frac{1}{EI_y}\left[-\left(1+\frac{\nu}{2}\right)\left(\frac{h^2}{4} - z^2\right)q(x) - M(x)\right], \tag{31}$$

which at the surface of the beam $w(x) = u_z(x, h/2)$ becomes

$$EI_y w_{,xx} = -M(x). \tag{32}$$

This equation is called the * Euler-Bernoulli beam equation*. By using $M_{,xx} = -q(x)$ we can write this as

$$(EI_y w_{,xx})_{,xx} = q(x), \tag{33}$$

in terms of the line load $q(x)$, or

$$EI_y w_{,xxxx} = q(x), \tag{34}$$

if EI_y does not depend on position x.

Buckling

Buckling is a sudden lateral deformation of a slender structural member that occurs under compressive load, resulting in a loss of load-carrying capacity. Unlike simple compression, where the member shortens uniformly, buckling involves a sudden transition to a bending mode when a critical load is exceeded. This section examines the buckling of slender beams under axial compression.

Problem Setup

Consider a slender beam of length L with uniform cross-section under uniform axial compression N (positive for compression). The beam is fixed at both ends or pinned at the ends (pin-pin boundary conditions). When the axial force exceeds a critical value, the beam becomes unstable and deforms laterally.

The governing equation combines the Euler-Bernoulli beam equation with the effect of axial compression. When the beam bends laterally by an amount $w(x)$, the axial force creates an additional bending moment (called the P-delta effect):

$$EI_y w_{,xxxx} + N w_{,xx} = 0 \tag{35}$$

This is the **buckling equation** for a beam under axial compression with no external distributed load.

Boundary Conditions

For a beam with pin-pin (simply supported) boundary conditions at both ends:

$$w(0) = 0 \tag{36}$$
$$w(L) = 0 \tag{37}$$
$$w_{,xx}(0) = 0 \tag{38}$$
$$w_{,xx}(L) = 0 \tag{39}$$

The first two conditions enforce zero displacement at the supports, while the last two enforce zero bending moment (free rotation).

Solution to the Buckling Equation

Let us rewrite Eq. (35) in the form:

$$w_{,xxxx} + \frac{N}{EI_y} w_{,xx} = 0 \tag{40}$$

Define $\beta^2 = \frac{N}{EI_y}$. Then:

$$w_{,xxxx} + \beta^2 w_{,xx} = 0 \tag{41}$$

This is a fourth-order linear ODE. Let $v(x) = w_{,xx}$, giving:

$$v_{,xx} + \beta^2 v = 0 \tag{42}$$

The general solution is:

$$v(x) = A\cos(\beta x) + B\sin(\beta x) \tag{43}$$

Integrating twice to find $w(x)$:

$$w(x) = C_1 + C_2 x + \frac{A}{\beta^2}\cos(\beta x) + \frac{B}{\beta^2}\sin(\beta x) \tag{44}$$

Applying the boundary condition $w(0) = 0$:

$$C_1 + \frac{A}{\beta^2} = 0 \implies C_1 = -\frac{A}{\beta^2} \tag{45}$$

Applying the boundary condition $w_{,xx}(0) = 0$:

$$-A\beta^2 = 0 \implies A = 0 \tag{46}$$

This gives $C_1 = 0$. The solution reduces to:

$$w(x) = C_2 x + \frac{B}{\beta^2}\sin(\beta x) \tag{47}$$

Applying the boundary condition $w(L) = 0$:

$$C_2 L + \frac{B}{\beta^2}\sin(\beta L) = 0 \tag{48}$$

Applying the boundary condition $w_{,xx}(L) = 0$:

$$w_{,xx}(x) = -B\beta^2\sin(\beta x) \tag{49}$$

At $x = L$:

$$-B\beta^2\sin(\beta L) = 0 \tag{50}$$

This equation is satisfied if either $B = 0$ (which gives the trivial solution $w(x) = 0$) or $\sin(\beta L) = 0$.

Euler Buckling Formula

The non-trivial buckling solutions occur when:

$$\sin(\beta L) = 0 \implies \beta L = n\pi, \quad n = 1, 2, 3, ... \tag{51}$$

Therefore:

$$\beta = \frac{n\pi}{L} \quad \text{and} \quad N = \beta^2 E I_y = \frac{n^2 \pi^2 E I_y}{L^2} \tag{52}$$

The **critical buckling load** (Euler buckling load) occurs at the lowest mode, $n = 1$:

$$\boxed{N_{\text{cr}} = \frac{\pi^2 E I_y}{L^2}} \tag{53}$$

This is the **Euler buckling formula** for a pin-pin beam. The buckling load is proportional to the flexural rigidity EI_y and inversely proportional to the square of the length L^2.

Physical Interpretation

1. **Slenderness ratio**: The critical load depends on L^2, meaning slender beams buckle at much lower loads than stocky beams. The slenderness ratio $\lambda = L/r$, where $r = \sqrt{I_y/A}$ is the radius of gyration, characterizes this effect.
2. **Buckling modes**: Higher modes correspond to buckling patterns with multiple half-waves. For $n = 2$, the beam develops two half-sine waves along its length, requiring twice the load of the first mode.
3. **Stability**: When $N < N_{\text{cr}}$, the beam is stable and returns to straight configuration if perturbed. When $N > N_{\text{cr}}$, the beam is unstable and buckles dramatically.

Effective Length

For boundary conditions other than pin-pin, the Euler formula is generalized:

$$N_{\text{cr}} = \frac{\pi^2 E I_y}{(KL)^2} \tag{54}$$

where K is the effective length factor:

- **Pin-pin**: $K = 1.0$ (ends are free to rotate)

- **Fixed-free**: $K = 2.0$ (one end fixed, one free)
- **Fixed-fixed**: $K = 0.5$ (both ends fixed)
- **Fixed-pin**: $K = 0.7$ (one end fixed, one pinned)

Higher K values correspond to longer effective lengths and lower buckling loads.

Buckling with Initial Imperfections

Real beams are never perfectly straight. With an initial deflection $w_0(x)$, the governing equation becomes:

$$EI_y w_{,xxxx} + N w_{,xx} = -N w_{0,xx} \tag{55}$$

This shows that initial imperfections are amplified by the axial load. The maximum deflection increases as N approaches the critical load, leading to failure through a gradual loss of stiffness rather than a sudden snap-through.

Chapter Summary

This chapter introduced plane stress and derived stresses in beams:

- **Plane stress**: $\sigma_{yy} = \tau_{xy} = \tau_{yz} = 0$ for thin structures; simplifies 3D to 2D analysis
- **Plane stress Hooke's law**: Strain-stress relations simplify when one normal stress vanishes
- **Beam stresses**: $\sigma_{xx} = Mz/I_y$ varies linearly through the cross-section (linear in z)
- **Shear stress**: $\tau_{xz} = Q(h^2/4 - z^2)/(2I_y)$ is parabolic through the thickness
- **Moment of inertia**: $I_y = \int z^2 \, dA$ is a geometric property of the cross-section
- **Euler-Bernoulli equation**: $EI_y w_{,xx} = -M(x)$ relates curvature to bending moment
- **Buckling equation**: $EI_y w_{,xxxx} + N w_{,xx} = 0$ governs stability under axial load
- **Euler formula**: $N_{\text{cr}} = \pi^2 EI_y/(KL)^2$ gives the critical buckling load
- **Effective length factor** K: Accounts for boundary conditions (pin-pin: $K = 1$, fixed-fixed: $K = 0.5$)
- **Initial imperfections**: Amplify deflections and reduce effective buckling capacity

Buckling is a critical failure mode for slender structures that can occur well below material yield stress.