Plates

Learning Objectives

After completing this chapter, you should be able to:

- **Extend** Euler-Bernoulli beam theory to plates by defining resultant forces (Q_x, Q_y) and moments (M_{xx}, M_{yy}, M_{xy})
- **Derive** the moment-curvature relations $M_{xx} = -K(w_{,xx} + \nu w_{,yy})$ with flexural rigidity $K = Eh^3/[12(1-\nu^2)]$
- **Apply** Kirchhoff's plate equation $\nabla^4 w = p/K$ to analyze plate deflection problems

Plates

Stress

Kirchhoff plate theory is the straightforward generalization of Euler-Bernoulli beam theory to plates. We abandon the plane situation in which all derivatives in y-direction vanish. The weak boundary conditions then become

$$Q_x(x,y) = \int_h \mathrm{d}z\, \tau_{xz}(x,y,z) \tag{1}$$

$$Q_y(x,y) = \int_h \mathrm{d}z\, \tau_{yz}(x,y,z) \tag{2}$$

$$M_{xx}(x,y) = \int_h \mathrm{d}z\, z\sigma_{xx}(x,y,z) \tag{3}$$

$$M_{yy}(x,y) = \int_h \mathrm{d}z\, z\sigma_{yy}(x,y,z) \tag{4}$$

$$M_{xy}(x,y) = \int_h \mathrm{d}z\, z\tau_{xy}(x,y,z), \tag{5}$$

where the integral is over the height h of the plate. Q_x and Q_y are called shear forces, M_{xx} and M_{yy} are bending moments and M_{xy} is the torsional moment.

Note that employing static equilibrium $\sigma_{ij,j} = 0$ we obtain

$$Q_{x,x} + Q_{y,y} = \int_{-h/2}^{h/2} \mathrm{d}z \, \left(\tau_{zx,x} + \tau_{zy,y}\right) = -\int_{-h/2}^{h/2} \mathrm{d}z \, \tau_{zz,z} \tag{6}$$

but

$$\int_{-h/2}^{h/2} \mathrm{d}z \, \tau_{zz,z} = \tau_{zz}(x,y,h/2) - \tau_{zz}(x,y,-h/2) \equiv p(x,y) \tag{7}$$

where $p(x,y)$ is the pressure on the plate (cf. also the corresponding equation Eq. (**??**) for the beam). Similarly

$$M_{xx,x} + M_{xy,y} = \int_{-h/2}^{h/2} \mathrm{d}z \, z \left(\tau_{xx,x} + \tau_{xy,y}\right) = -\int_{-h/2}^{h/2} \mathrm{d}z \, z \tau_{xz,z} \tag{8}$$

and

$$\int_{-h/2}^{h/2} \mathrm{d}z \, z \tau_{xz,z} = \left[z \tau_{xz}\right]_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \mathrm{d}z \, \tau_{xz} = -Q_x. \tag{9}$$

where the last equality holds because $\tau_{xz}(x,y,h/2) = -\tau_{xz}(x,y,-h/2)$. The condition for static equilibrium $\sigma_{ij,j} = 0$ therefore becomes

$$Q_{x,x} + Q_{y,y} = -p(x,y) \tag{10}$$

$$M_{xx,x} + M_{xy,y} = Q_x(x,y) \tag{11}$$

$$M_{xy,x} + M_{yy,y} = Q_y(x,y) \tag{12}$$

in the weak form. Note that this can be written in the compact form $Q_{i,i} = -p$ and $M_{ij,j} = Q_i$.

As in the Euler-Bernoulli case, we assume that the components σ_{xx}, σ_{yy} and τ_{xy} vary linearly with z. We can write

$$\sigma_{xx}(x,y,z) = \frac{M_{xx}(x,y)}{I} z \tag{13}$$

$$\sigma_{yy}(x,y,z) = \frac{M_{yy}(x,y)}{I} z \tag{14}$$

$$\tau_{xy}(x,y,z) = \frac{M_{xy}(x,y)}{I} z \tag{15}$$

with $I = \int dz \, z^2 = h^3/12$. The remaining components of the stress tensor are obtained from static equilibrium. Static equilibrium yields

$$\tau_{xz,z} = -\frac{z}{I} Q_x \quad \text{and} \quad \tau_{yz,z} = -\frac{z}{I} Q_y \tag{16}$$

which can be integrated under the condition $\tau_{xz}(x,y,h/2) = \tau_{xz}(x,y,-h/2) = 0$ to

$$\tau_{xz}(x,y,z) = \frac{Q_x}{2I}\left(\frac{h^2}{4} - z^2\right) \quad \text{and} \quad \tau_{yz}(x,y,z) = \frac{Q_y}{2I}\left(\frac{h^2}{4} - z^2\right). \tag{17}$$

This is analogous to Eq. (**??**) for the beam.

We are finally left with finding an expression for σ_{zz}. Again we use static equilibrium to obtain

$$\sigma_{zz,z} = -\tau_{xz,x} - \tau_{yz,y} = \frac{p(x,y)}{2I}\left(\frac{h^2}{4} - z^2\right). \tag{18}$$

Integration under the condition that the loads on top and bottom surface of the plate balance, $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2)$, gives

$$\sigma_{zz}(x,y,z) = \frac{p(x,y)}{2I}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) z. \tag{19}$$

At the top and bottom of the plate we find $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2) = p(x, y)/2$.

Displacements

Now that we know the stress inside the plate, we can again compute the displacements from Hooke's law. In the full three-dimensional case, Hooke's law,

$$\varepsilon_{xx} \equiv u_{x,x} = (\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz})/E \tag{20}$$

$$\varepsilon_{yy} \equiv u_{y,y} = (\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz})/E \tag{21}$$

$$2\varepsilon_{xz} \equiv u_{x,z} + u_{z,x} = 2(1+\nu)\tau_{xz}/E \tag{22}$$

$$2\varepsilon_{yz} \equiv u_{y,z} + u_{z,y} = 2(1+\nu)\tau_{yz}/E \tag{23}$$

$$2\varepsilon_{xy} \equiv u_{x,y} + u_{y,x} = 2(1+\nu)\tau_{xy}/E, \tag{24}$$

and taking the derivative of Eq. (22) with respect to x and of Eq. (23) with respect to y, we obtain

$$u_{x,xz} + u_{z,xx} = 2(1+\nu)\tau_{xz,x}/E \tag{25}$$

$$u_{y,yz} + u_{z,yy} = 2(1+\nu)\tau_{yz,y}/E \tag{26}$$

and

$$u_{x,xz} + u_{z,xx} = \partial_z(u_{x,x}) + u_{z,xx} = u_{z,xx} + (\sigma_{xx,z} - \nu\sigma_{yy,z} - \nu\sigma_{zz,z})/E \tag{27}$$

$$u_{y,yz} + u_{z,yy} = \partial_z(u_{y,y}) + u_{z,yy} = u_{z,yy} + (\sigma_{yy,z} - \nu\sigma_{xx,z} - \nu\sigma_{zz,z})/E. \tag{28}$$

Combining Eqs. (25), (27) and Eqs. (26), (28) and noting that $\sigma_{zz,z} = -\tau_{xz,x} - \tau_{xz,y}$ yields

$$u_{z,xx} = \left[(2+\nu)\tau_{xz,x} - \nu\tau_{yz,y} - \sigma_{xx,z} + \nu\sigma_{yy,z}\right]/E \tag{29}$$

$$u_{z,yy} = \left[(2+\nu)\tau_{yz,y} - \nu\tau_{xz,x} - \sigma_{yy,z} + \nu\sigma_{xx,z}\right]/E. \tag{30}$$

We now create linear combination of these expressions such that $\sigma_{xx,z} = M_{xx}/I$ or $\sigma_{yy,z} = M_{yy}/I$ drop out,

$$u_{z,xx} + \nu u_{z,yy} = \left[(2+\nu-\nu^2)\tau_{xz,x} - \nu(1+\nu)\tau_{yz,y} - (1-\nu^2)M_{xx}/I\right]/E \quad (31)$$
$$u_{z,yy} + \nu u_{z,xx} = \left[(2+\nu-\nu^2)\tau_{yz,y} - \nu(1+\nu)\tau_{xz,x} - (1-\nu^2)M_{yy}/I\right]/E. \quad (32)$$

We now only consider the displacement at the surface, $w(x, y) \equiv u_z(x, y, h/2)$. Since the surfaces are traction free, all terms involving τ_{xz} and τ_{yz} vanish. Hence

$$M_{xx} = -K(w_{,xx} + \nu w_{,yy}) \quad (33)$$
$$M_{yy} = -K(w_{,yy} + \nu w_{,xx}) \quad (34)$$

with the *flexural rigidity* $K = EI/(1-\nu^2) = Eh^3/[12(1-\nu^2)]$.

Finally, we are looking for an expression for $M_{xy} = I\tau_{xy,z}$. We have from Eqs. (22)-(24)

$$\frac{2(1+\nu)}{EI}M_{xy} = u_{x,yz} + u_{y,xz} = \frac{2(1+\nu)}{E}(\tau_{xz,y} + \tau_{yz,x}) - 2u_{z,xy}, \quad (35)$$

which yields

$$M_{xy} = -K(1-\nu)w_{,xy}, \quad (36)$$

the desired expression.

We now plug Eqs. (33), (34) and (36) into the equilibrium conditions Eqs. (11) and (12). This yields

$$-K(w_{,xxx} + w_{,xyy}) = Q_x(x, y) \quad (37)$$
$$-K(w_{,yyy} + w_{,xxy}) = Q_y(x, y) \quad (38)$$
$$-K(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) = -p(x, y). \quad (39)$$

The last expression is Kirchhoff's equation,

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} = \nabla^2\nabla^2 w = \nabla^4 w = \frac{p}{K}, \quad (40)$$

that governs the deformation of plates.

Chapter Summary

This chapter extended beam theory to 2D plate problems:

- **Kirchhoff plate theory**: Generalizes Euler-Bernoulli beam theory to two dimensions
- **Resultant quantities**: Shear forces Q_x, Q_y and moments M_{xx}, M_{yy}, M_{xy} are

integrals through thickness

- **Equilibrium**: $Q_{i,i} = -p$ and $M_{ij,j} = Q_i$ in weak form
- **Stress distributions**: σ_{xx}, σ_{yy}, τ_{xy} linear in z; τ_{xz}, τ_{yz} parabolic in z
- **Moment-curvature**: $M_{xx} = -K(w_{,xx} + \nu w_{,yy})$ with flexural rigidity $K = Eh^3/[12(1-\nu^2)]$
- **Kirchhoff equation**: $\nabla^4 w = p/K$ is the biharmonic governing equation for plate deflection
- **Biharmonic operator**: $\nabla^4 = \nabla^2\nabla^2$ appears in both plate bending and fracture mechanics

Plate theory is essential for analyzing thin structural elements like floors, panels, and MEMS devices.

Bibliography