Plastic failure

Learning Objectives

After completing this chapter, you should be able to:

- **Calculate** stress invariants (I_1, J_2) and principal stresses from a given stress tensor
- **Apply** the von Mises yield criterion $\sigma_e = \sqrt{3J_2} = \sigma_y$ to predict plastic failure
- **Use** the Drucker-Prager criterion for pressure-dependent materials like soils and concrete

Plastic failure occurs when a material transitions from elastic to plastic (inelastic) behavior. At the point of transition, the material begins to deform permanently under stress. To predict when this transition occurs, we use **yield criteria**, which are mathematical conditions defining the limit of elastic behavior in terms of the applied stress state.

Stress invariants

Note

We have encountered vectors in the first chapters on statics of rigid bodies and have intuitively worked with them. To recap, a vector is an object that represents a direction and a magnitude. Geometrically, they are often represented as arrows. In a Cartesian coordinate system (or basis), a vector can be represented by a set of numbers. For example in two dimensions, we denote the components of the vector $\vec{v}$ as the x and y components and typically write the vector in the column-form

$$\vec{v} = v_x \hat{x} + v_y \hat{y} \equiv \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{1}$$

where v_x and v_y are real numbers that are called the * components * of the vector $\vec{v}$. $\hat{x}$ and $\hat{y}$ are vectors of unit length that point along the x- and y-directions of the coordinates. The length of the vector is given by the 2-norm $|\vec{v}| \equiv \|\vec{v}\|_2 = (v_x^2 + v_y^2)^{1/2}$. It is important to emphasize that this is a * representation * of the vector and the components $v_i \in \mathbb{R}$ depend on the specific basis $\hat{x}$ and $\hat{y}$. This representation is called a * tensor of order 1*. The order of a tensor is sometimes also called degree or rank. Any component-wise representation, such as the one on the right hand side of Eq. (1), implies a basis. The basis is written explicitly as $\hat{x}$ and $\hat{y}$ in the middle expression of Eq. (1). Note that in component notation, the basis vectors are*

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

We will restrict the discussion here to orthogonal bases where $\hat{x} \cdot \hat{y} = 0$. The discussion here will use two-dimensional examples, but a generalization to more that two dimensions is straightforward Formally, a vector is an element of a vector space. *A vector space V (often also called a* linear space*) is a set of objects (for example the set containing our basis vectors $\hat{x}$ and $\hat{y}$ and linear combinations thereof) along with two operations: Addition (of two vectors) and multiplication (of a vector) with a scalar. These operations again yield a vector, i.e. an element of V. This can be expressed as

- $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$
- $a \in \mathbb{R}$ and $\vec{u} \in V$, then $a \cdot \vec{u} \in V$ and hence Eq. (1) yields a vector. In general, we may multiply the vectors by an element of a * algebraic number field *$\mathbb{F}$ rather than $\mathbb{R}$. Then we say V is a vector space over the field $\mathbb{F}$. In these notes (and our lectures) we will always deal with either real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) numbers.

Recall the concept of an algebraic number field in mathematics. A field is an algebraic structure that is a set along with two operations "+" and "$ $" associating an element with two elements of the set. The operations are required to satisfy the field axioms: - Associativity of addition and multiplication: $a+(b+c) = (a+b)+c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

- Commutativity of addition and multiplication: $a + b = b + a$ and $a \cdot b = b \cdot a$.
- Additive and multiplicative identity: $0 \in \mathbb{F}$ with $a + 0 = a$ and $1 \in \mathbb{F}$ with $1 \cdot a = a$.
- Additive inverses: $\forall a \in \mathbb{F}$ we have an inverse element $i = -a$ with $a+i = a+(-a) = 0$
- Multiplicative inverses: $\forall a \neq 0$ we have an element a^{-1} with $a \cdot a^{-1} = 1$.
- Distributivity of multiplication over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ Note that in physics a * field * is typically a quantity that depends on position and not an algebraic number field. A scalar field $\phi(\vec{r})$ would be scalar quantity $\phi \in \mathbb{R}$ that depends on a vector (the position) $\vec{r} \in V$. While these two meanings of the term * field * exist, we will always refer to this latter physical meaning in the following.

Additionally, we have here used the dot $ $ to indicate a scalar multiplication, but we will reserve this dot in the rest of these notes for contractions, i.e. operations that implicitly include a sum.

The simplest (and also most important) operation on a vector is a linear transformation. The simplest linear transformation is the multiplication with a scalar $a \in \mathbb{R}$. In terms of the geometric interpretation of a vector, this multiplication would change the length of the vector by a but not its direction.
A general linear transformation can in addition change the direction of the vector and therefore represent for example rotations. Given a vector $\vec{u}$ and a scalar $ $, a linear transformation $\mathcal{L}$ to a vector $\vec{u}' = \mathcal{L}\vec{u}$ has the properties:

$$\mathcal{L}(\alpha\vec{u}) = \alpha\mathcal{L}\vec{u} \tag{3}$$

$$\mathcal{L}(\vec{u}+\vec{v}) = \mathcal{L}\vec{u} + \mathcal{L}\vec{v} \tag{4}$$

We will only deal with linear operations that map between the same vector space V, $\mathcal{L} : V \mapsto V$. (It may map between quantities with different physical units.) Given a component-wise (Cartesian) representation of a vector, Eq. (1), a linear transformation can be expressed as a multiplication by a matrix. This is easily by applying Eqs. (3) and (4) to

$$\vec{w} = \mathcal{L}\vec{v} = \mathcal{L}(v_x\hat{x} + v_y\hat{y}) = v_x\mathcal{L}\hat{x} + v_y\mathcal{L}\hat{y}. \tag{5}$$

We can express any element of our vector space as a linear combination of the basis vectors, hence also

$$\mathcal{L}\hat{x} = L_{xx}\hat{x} + L_{yx}\hat{y} \tag{6}$$

$$\mathcal{L}\hat{y} = L_{xy}\hat{x} + L_{yy}\hat{y}. \tag{7}$$

Application of the linear operation to an arbitrary vector, Eq. (5), can therefore be expressed as

$$\vec{w} = \mathcal{L}\vec{v} = (L_{xx}v_x + L_{xy}v_y)\hat{x} + (L_{yx}v_x + L_{yy}v_y)\hat{y} \equiv \underline{L} \cdot \vec{v} \tag{8}$$

where $\underline{L} \cdot \vec{v}$ is the multiplication of the matrix $\underline{L}$ with the vector $\vec{v}$. A matrix is therefore a * representation * of a linear operation. Note that from Eq. (6) it is straightforward to see that formally we obtain the components of the matrix from

$$L_{ij} = \hat{i} \cdot \mathcal{L}\hat{j}. \tag{9}$$

Here the $ $ indicates the inner product or contraction of two vectors. In component-wise notation we can express Eq. (8) as

$$w_i = \sum_{j=x,y} L_{ij}v_j \equiv L_{ij}v_j \tag{10}$$

where the last term on the right-hand side uses the * Einstein summation * convention. In this convention, a summation over repeated indices within the same quantity or in products is implicit. This summation is also often called a * contraction * and in dyadic notation it is indicated by a centered dot, e.g. $\vec{w} = \underline{L} \cdot \vec{v}$. It is straighforward to "convert" from dyadic notation to component-wise or * index notation. *Imagine the i, j component of the resultant matrix of the product*

$$[\underline{A} \cdot \underline{B} \cdot \underline{C}]_{ij} = A_{ik} B_{kl} C_{lj}. \tag{11}$$

Converting from the dyadic notation to index notation involves identifying the indices of the resultant (first index of the first matrix in the product i and last index of the last matrix in the product j) and introducing repeated indices for the summation. These indices, k and l in the example, always sit next to each other and are the indices that are contracted*. The advantage of the index notation is that it is unambiguous, but it may hide the physical structure of the underlying operations. In these notes, we will therefore intermix "dyadic" notation as in Eq. (8) and index notation as appropriate. Note that there is also a double contraction, for example

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \tag{12}$$

that in dyadic notation we would indicate with two dots: $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$. Each dot therefore represents a sum over some (hidden) index.

Rotating the stress tensor

A tensor is a representation of a linear transformation: A tensor of order 2 (a matrix) transforms a tensor of order 1 (a vector) into another tensor of order 1. We have already encountered the Cauchy stress tensor $\underline{\sigma}$ of solid mechanics. It transforms an area vector $\vec{A}$ into a force vector $\vec{F}$,

$$\vec{F} = \underline{\sigma} \cdot \vec{A} \quad \text{or in index notation} \quad F_i = \sigma_{ij} A_j. \tag{13}$$

We describe $\vec{F}$, $\vec{A}$ and $\underline{\sigma}$ in terms of their components, i.e. we describe a realization of the linear transformation (and the area and force vectors).

A rotation does not affect the effect of the linear transformation. Hence, if we change the coordinate system from $\hat{x}$, $\hat{y}$ to $\hat{x}'$, $\hat{y}'$, then the component of our vector $\vec{F}$ change to

$$F'_x = \vec{F} \cdot \hat{x}' = F_x \hat{x} \cdot \hat{x}' + F_y \hat{y} \cdot \hat{x}' \tag{14}$$

$$F'_y = \vec{F} \cdot \hat{y}' = F_y \hat{x} \cdot \hat{y}' + F_y \hat{y} \cdot \hat{y}'. \tag{15}$$

Comparing with Eq. (**??**) yields the compact notation

$$\vec{F}' = \underline{R}^T \cdot \vec{F} \quad \text{or in index notation} \quad F'_i = F_j R_{ji}. \tag{16}$$

Note that the transpose shows up because $\underline{R}$ describes the rotation of the basis and we are here rotating a vector expressed within this basis, which is the inverse operation.

Since we now understand how to rotate vectors, we can ask the question of how to rotate a tensor of order 2. Starting from Eq. (13) and using the inverse of Eq. (16), we write

$$\underline{R} \cdot \vec{F}' = \underline{\sigma} \cdot \left(\underline{R} \cdot \vec{A}' \right), \tag{17}$$

and multiply by $\underline{R}^T$ from the left to yield

$$\vec{F}' = \left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) \cdot \vec{A}'. \tag{18}$$

In the rotated coordinate system, the tensor attains the representation

$$\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \quad \text{or in index notation} \quad \sigma'_{ij} = \sigma_{kl} R_{ki} R_{lj} \tag{19}$$

since this leave the expression for the linear transformation

$$\vec{F}' = \underline{\sigma}' \cdot \vec{A}'. \tag{20}$$

invariant. The trace and determinant of the rotated tensor are

$$\text{tr}\underline{\sigma}' = \text{tr}\left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) = \text{tr}\left(\underline{R} \cdot \underline{R}^T \cdot \underline{\sigma} \right) = \text{tr}\underline{\sigma} \tag{21}$$

$$\det \underline{\sigma}' = \det \left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) = \det \left(\underline{R} \cdot \underline{R}^T \cdot \underline{\sigma} \right) = \det \underline{\sigma} \tag{22}$$

and hence invariant under rotation. Note that in general for an $n \times n$ tensor, there are n invariants; more on this will be discussed below when talking about eigenvalues.

Since we now understand how to rotate tensors of order 2, we can ask the question how to rotate a tensor of order 4. As an example, we use the stiffness tensor $\underline{\underline{C}}$, that we will encounter in the next chapter. This tensor transforms a strain tensor $\underline{\varepsilon}$ into a stress tensor $\underline{\sigma}$,

$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon} \quad \text{or in index notation} \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl}. \tag{23}$$

We can rewrite this using the inverse of Eq. (19) as

$$\underline{R} \cdot \underline{\sigma}' \cdot \underline{R}^T = \underline{\underline{C}} : \left(\underline{R} \cdot \underline{\varepsilon}' \cdot \underline{R}^T \right) \quad \text{or} \quad R_{ik} R_{jl} \sigma'_{kl} = C_{ijkl} R_{km} R_{ln} \varepsilon'_{mn}, \tag{24}$$

which we now multiply from the left with $\underline{R}^T$ and from the right with $\underline{R}$. This gives

$$\sigma'_{ij} = C_{mnop} R_{mi} R_{nj} R_{ok} R_{pl} \varepsilon'_{kl}, \tag{25}$$

and hence the transformation rule

$$C'_{ijkl} = C_{mnop} R_{mi} R_{nj} R_{ok} R_{pl}. \tag{26}$$

Quantities that transform as Eqs. (16), (19) and (26) are called * tensors*.

Principal stresses

Let us discuss in more detail what happens if we rotate a symmetric tensor of order 2, i.e. a tensor that fulfills $\underline{\sigma}^T = \underline{\sigma}$. From Eq. (19) it is straightforward to see, that $\underline{\sigma}'^T = \underline{\sigma}'$, i.e. the transformed tensor is also symmetric.

We now explicitly write the rotation for the tensor

$$\underline{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}, \tag{27}$$

using the rotation

matrix Eq. (**??**). This gives symmetric $\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R}$ with the components

$$\sigma'_{xx} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \cos 2\theta + \sigma_{xy} \sin 2\theta \tag{28}$$

$$\sigma'_{yy} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \cos 2\theta - \sigma_{xy} \sin 2\theta \tag{29}$$

$$\sigma'_{xy} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \sin 2\theta + \sigma_{xy} \cos 2\theta. \tag{30}$$

There is a specific rotation angle θ_0 where the diagonal elements σ'_{xx} and σ'_{yy} become extremal. It is determined from $\sigma'_{xx,\theta} = \sigma'_{yy,\theta} = 0$,

$$\tan 2\theta_0 = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}. \tag{31}$$

At this rotation angle we find that the off-diagonal components vanish, $\sigma'_{xy}(\theta_0) = 0$, and the rotated matrix is diagonal,

$$\underline{\sigma} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \tag{32}$$

with diagonal elements

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right]^{1/2} \tag{33}$$

$$\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right]^{1/2}. \tag{34}$$

This is the simplest example of the diagonalization of a matrix. The elements of the diagonalized Cauchy stress tensor, σ_1 and σ_2, are called the * principal stresses*.

The diagonalization of a symmetric matrix always corresponds to the rotation into a new coordinate system. We have explictly shown this for the two-dimensional case here and will now show it in more generality for the three-dimensional case. In this process, we will encounter the concept of a stress invariant.

Stress invariants

Equation (32) fulfills the eigenvalue equations $\underline{\sigma}\cdot\hat{x} = \sigma_1\hat{x}$ and $\underline{\sigma}\cdot\hat{y} = \sigma_2\hat{y}$. Rather than explicitly computing a rotation, we can ask the question whether we can find a scalar $ $ and a vector $\vec{v}$ that fulfills the eigenvalue equation

$$\underline{\sigma}\cdot\vec{v} = \lambda\vec{v}. \tag{35}$$

This equation of course has the trivial solution $\vec{v} = 0$. It can only have a nontrivial solution if

$$\det\left(\underline{\sigma} - \lambda\underline{1}\right) = 0. \tag{36}$$

For a $n \times n$ matrix, Eq. (36) leads to a polynomial of order n in $ $ with n (possibly complex valued) solutions.

For the case of a symmetric 3×3 matrix, we can write this down explicitly as

$$\det\begin{pmatrix} \sigma_{xx}-\lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\lambda \end{pmatrix} = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0 \tag{37}$$

with

$$I_1 = \text{tr}\underline{\sigma} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{38}$$

$$I_2 = \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} + \sigma_{xx}\sigma_{yy} - \sigma_{yz}^2 - \sigma_{xz}^2 - \sigma_{xy}^2 \tag{39}$$

$$I_3 = \det\underline{\sigma} \tag{40}$$

The quantities I_1 to I_3 are called * invariants*. We have shown above explicitly that the trace and the determinant are invariant under rotation. The same holds true for all coefficients of the characteristic polynomial (two of which are actually trace and determinant). This is because

$$\det\left(\underline{R}^T\cdot\underline{\sigma}\cdot\underline{R} - \lambda\underline{1}\right) = \det\left[\underline{R}^T\cdot\left(\underline{\sigma} - \lambda\underline{1}\right)\cdot\underline{R}\right] = \det\left(\underline{\sigma} - \lambda\underline{1}\right). \tag{41}$$

The 3-dimensional tensor therefore has three invariants. These invariants have important physical interpretations. For the stress tensor, I_1 is related to the hydrostatic stress and I_2 to the shear stress.

Note that for a diagonal matrix,

$$\underline{\sigma}' = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \tag{42}$$

the invariants are

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{43}$$

$$I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \tag{44}$$

$$I_3 = \sigma_1\sigma_2\sigma_3. \tag{45}$$

By equating Eqs. (38) to (40) with Eqs. (43) to (45) we can calculate the eigenvalues σ_1, σ_2 and σ_3.

Once we have computed the eigenvalues, we can obtain the corresponding eigenvectors by solving

$$\underline{\sigma}\vec{v}_1 = \sigma_1\vec{v}_1, \quad \underline{\sigma}\vec{v}_2 = \sigma_2\vec{v}_2, \quad \text{and} \quad \underline{\sigma}\vec{v}_3 = \sigma_3\vec{v}_3. \tag{46}$$

Note that the expressions only determine the direction of

$\vec{v}_i$, not its length, and we are free to require $|\vec{v}_i| = 1$. Furthermore, let us regard scalar products $\vec{v}_1 \cdot \vec{v}_2$, then

$$\sigma_1\vec{v}_1 \cdot \vec{v}_2 = (\underline{\sigma} \cdot \vec{v}_1) \cdot \vec{v}_2 = \vec{v}_1 \cdot (\underline{\sigma}^T \cdot \vec{v}_2) = \vec{v}_1 \cdot (\underline{\sigma} \cdot \vec{v}_2) = \sigma_2\vec{v}_1 \cdot \vec{v}_2 \tag{47}$$

and if $\sigma_1 \neq \sigma_2$ we must have $\vec{v}_1 \cdot \vec{v}_2 = 0$. Hence the eigenvectors of a symmetric matrix are orthonormal, or in other words, they form the basis of a coordinate system.

We can write Eq. (46) in the more compact notation

$$\underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \cdot \underline{R} \tag{48}$$

with $\underline{R} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Multiplying from the left with $\underline{R}^{-1} = \underline{R}^T$ (this holds because the eigenvectors are orthonormal) we get

$$\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}. \tag{49}$$

This is nothing else than a coordinate transformation (rotation) of the tensor $\underline{\sigma}$. Since the diagonalization of a symmetric matrix leads to orthonormal eigenvectors, the diagonalization is a rotation of the coordinate system.

Hydrostatic and deviatoric stress

We call $\sigma_h = \mathrm{tr}\underline{\sigma}/3 = I_1/3$ the * hydrostatic stress*. It is the stress measure that tells us about volume expansion and contraction. The pressure is the negative of the hydrostatic stress, $p = -\sigma_h$. Our sign convention is such that positive stresses (negative pressures) mean tension and negative stresses (positive pressures) compression.

Using the hydrostatic stress, we can construct yet another stress tensor that quantifies the * deviation * from a pure hydrostatic condition with stress $\sigma_h\underline{1}$. We define

$$\underline{s} = \underline{\sigma} - \sigma_h\underline{1}, \tag{50}$$

the * deviatoric stress*. Note that this tensor is constructed such that $\mathrm{tr}\underline{s} = 0$.

The invariants of the deviatoric stress are commonly denoted by the symbol J. We already know that $J_1 = 0$ by construction. The second invariant is given by

$$\begin{aligned} J_2 &= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right] \\ &= \frac{1}{6}\left[(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{xx}-\sigma_{zz})^2+6(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{xz}^2)\right]. \end{aligned} \tag{51}$$

From this invariant we can derive the * von-Mises stress *$\sigma_{vM} = \sqrt{3J_2}$. The von-Mises stress characterizes the pure shear contribution to the stress. The second invariant of the deviatoric stress plays an important role in plasticity models, where it is often assumed that a material flows when the von-Moses stress exceeds a certain threshold.

Example: Principal stresses from stress tensor

Tip

Consider a 2D stress state in the xy-plane:

$$\underline{\sigma} = \begin{pmatrix} 80 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ MPa}$$

Find:

- The principal stresses $\sigma_1, \sigma_2, \sigma_3$
- The maximum shear stress $\tau_{\max}$

Solution:
The given stress tensor describes a 2D stress state: there are normal stresses $\sigma_{xx} = 80$ MPa and $\sigma_{yy} = 20$ MPa, plus a shear stress $\tau_{xy} = 30$ MPa. The zeros in the third row and column indicate no stress components involving the z-direction (plane stress condition). Our goal is to find the *principal stresses*—the normal stresses that would act on specially oriented planes where the shear stress vanishes. These are the eigenvalues of the stress tensor.
Principal stresses are found by solving the eigenvalue equation $\det(\underline{\sigma} - \sigma\underline{I}) = 0$. Since the third row and column are all zeros except for the diagonal, the z-direction is already a principal direction with $\sigma_3 = 0$. We only need to find the eigenvalues of the 2D submatrix:

$$\underline{\sigma}_{2D} = \begin{pmatrix} 80 & 30 \\ 30 & 20 \end{pmatrix}$$

The eigenvalue equation becomes:

$$\det\begin{pmatrix} 80-\sigma & 30 \\ 30 & 20-\sigma \end{pmatrix} = 0$$

For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $ad - bc$. Applying this:

$$(80 - \sigma)(20 - \sigma) - (30)(30) = 0$$

Expanding the first term:

$$(80 - \sigma)(20 - \sigma) = 80 \cdot 20 - 80\sigma - 20\sigma + \sigma^2 = 1600 - 100\sigma + \sigma^2$$

Subtracting $30^2 = 900$:

$$\sigma^2 - 100\sigma + 1600 - 900 = 0$$
$$\sigma^2 - 100\sigma + 700 = 0$$

This is the *characteristic equation* of the stress tensor. We solve it using the quadratic formula: for $a\sigma^2 + b\sigma + c = 0$, the solutions are $\sigma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Here $a = 1$, $b = -100$, $c = 700$:

$$\sigma = \frac{100 \pm \sqrt{(-100)^2 - 4(1)(700)}}{2(1)} = \frac{100 \pm \sqrt{10000 - 2800}}{2} = \frac{100 \pm \sqrt{7200}}{2}$$

Computing $\sqrt{7200} = \sqrt{3600 \cdot 2} = 60\sqrt{2} \approx 84.85$:

$$\sigma = \frac{100 \pm 84.85}{2}$$

This gives two solutions:

$$\sigma_1 = \frac{100 + 84.85}{2} = \frac{184.85}{2} = 92.43 \text{ MPa}$$
$$\sigma_2 = \frac{100 - 84.85}{2} = \frac{15.15}{2} = 7.57 \text{ MPa}$$

By convention, we label principal stresses such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. So we have $\sigma_1 = 92.43$ MPa (largest, tensile), $\sigma_2 = 7.57$ MPa (intermediate, tensile), and $\sigma_3 = 0$ MPa (smallest, from the z-direction).

The maximum shear stress occurs on planes oriented at 45° to the principal directions. Its magnitude is:

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{\sigma_1 - \sigma_3}{2} = \frac{92.43 - 0}{2} = 46.21 \text{ MPa}$$

Note that we use σ_1 and σ_3 (not σ_2) because the maximum shear stress depends on the difference between the *largest* and *smallest* principal stresses.

The stress tensor has been transformed to principal coordinates where the largest tensile stress is 92.43 MPa (acting in a direction rotated from the original x-axis), a moderate tensile stress of 7.57 MPa acts perpendicular to it, no stress acts perpendicular to the xy-plane (this is the plane stress condition), and the maximum shear stress is 46.21 MPa (acting at ±45° to the principal directions).

We can verify our answer using the invariants. The trace (sum of eigenvalues) should equal $\sigma_1 + \sigma_2 + \sigma_3 = 92.43 + 7.57 + 0 = 100$ MPa, which matches $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 80 + 20 + 0 = 100$ MPa.

Introduction to Yield Criteria

A yield criterion is a function $F(\underline{\sigma})$ of the stress tensor that defines a surface in stress space. When the material is loaded elastically (inside or on the yield surface), $F(\underline{\sigma}) < 0$ or $F(\underline{\sigma}) = 0$. When the stress exceeds the yield surface, $F(\underline{\sigma}) > 0$, and the material begins to deform plastically.

The most important yield criteria are:

1. **Tresca (Maximum Shear Stress)**: Simple but less commonly used
2. **Von Mises (von Mises-Huber-Hencky)**: Most widely used for metals
3. **Drucker-Prager**: Accounts for mean stress effects, useful for granular and geotechnical materials

Stress Invariants

Before discussing yield criteria, we need to introduce **stress invariants**, which are scalar quantities derived from the stress tensor that remain unchanged under coordinate rotation. The three principal invariants of the stress tensor are:

$$I_1 = \text{tr}(\underline{\sigma}) = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 \tag{52}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses.

$$I_2 = \frac{1}{2}\left[(\text{tr}\underline{\sigma})^2 - \text{tr}(\underline{\sigma}^2)\right] \tag{53}$$

$$I_3 = \det(\underline{\sigma}) \tag{54}$$

The **mean stress** (hydrostatic pressure) is:

$$\sigma_m = \frac{I_1}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{55}$$

The **deviatoric stress tensor** represents the stress deviations from the mean stress:

$$\underline{s} = \underline{\sigma} - \sigma_m \underline{1} \tag{56}$$

where $\underline{1}$ is the identity matrix.

The **deviatoric stress invariant** (second invariant of deviatoric stress) is:

$$J_2 = \frac{1}{2}\text{tr}(\underline{s}^2) = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{57}$$

The **equivalent stress** (or von Mises stress) is defined as:

$$\sigma_e = \sqrt{3J_2} \tag{58}$$

Von Mises Yield Criterion

The **von Mises yield criterion** is the most commonly used criterion for metals and states that plastic flow initiates when the deviatoric stress reaches a critical value:

$$\sigma_e = \sigma_y \quad \text{or} \quad \sqrt{3J_2} = \sigma_y \tag{59}$$

where σ_y is the **yield stress** (uniaxial).

Equivalently, this can be written as:

$$\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{2}\sigma_y \tag{60}$$

In terms of the full stress tensor components:

$$\begin{aligned}\sigma_e = &\sqrt{\frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2\right]} \\ &+ 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\end{aligned} \tag{61}$$

Key characteristics of the von Mises criterion:

1. **Pressure independent**: The yield surface does not depend on the mean stress σ_m. This means that hydrostatic pressure alone (without shear) does not cause plastic flow.
2. **Cylindrical symmetry**: The yield surface in principal stress space is a cylinder aligned with the hydrostatic axis ($\sigma_1 = \sigma_2 = \sigma_3$).
3. **Isotropic**: The criterion applies equally in all directions, suitable for isotropic materials like metals.

Uniaxial tension: For a uniaxial stress state $\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$:

$$\sigma_e = \sigma = \sigma_y$$

Pure shear: For pure shear $\sigma_1 = \tau$, $\sigma_2 = 0$, $\sigma_3 = -\tau$:

$$\sigma_e = \sqrt{3}\tau = \sigma_y \implies \tau = \frac{\sigma_y}{\sqrt{3}}$$

Drucker-Prager Yield Criterion

The **Drucker-Prager yield criterion** generalizes the von Mises criterion to account for the effect of mean stress, making it particularly useful for materials like soil, concrete, and rock where the yield stress depends on hydrostatic pressure.

The Drucker-Prager yield criterion is defined as:

$$F(\underline{\sigma}) = \sqrt{J_2} + \alpha I_1 - k = 0 \tag{62}$$

or equivalently:

$$\sqrt{J_2} + \alpha\sigma_m - k = 0 \tag{63}$$

where α and k are material parameters that must be determined from experiments.

Physical interpretation:

- $\sqrt{J_2}$ represents the magnitude of deviatoric stress (similar to von Mises)
- αI_1 (or $\alpha\sigma_m$) represents the pressure-dependent component
- When $\alpha = 0$, the criterion reduces to von Mises
- When $\alpha > 0$, the yield stress increases with hydrostatic pressure (typical for granular materials)

Determining material parameters:

The parameters α and k are typically determined from two simple tests:

1. **Uniaxial compression**: $\sigma_1 = \sigma_c$ (compression), $\sigma_2 = \sigma_3 = 0$

   At yield: $I_1 = -\sigma_c$, $J_2 = \frac{\sigma_c^2}{3}$

$$\frac{\sigma_c}{\sqrt{3}} - \alpha\sigma_c - k = 0$$

2. **Uniaxial tension**: $\sigma_1 = \sigma_t$ (tension), $\sigma_2 = \sigma_3 = 0$

   At yield: $I_1 = \sigma_t$, $J_2 = \frac{\sigma_t^2}{3}$

$$\frac{\sigma_t}{\sqrt{3}} + \alpha\sigma_t - k = 0$$

From these two equations, we can solve for α and k:

$$\alpha = \frac{1}{2}\left(\frac{1}{\sqrt{3}\sigma_t} - \frac{1}{\sqrt{3}\sigma_c}\right) = \frac{\sigma_c - \sigma_t}{2\sqrt{3}\sigma_t\sigma_c} \tag{64}$$

$$k = \frac{1}{\sqrt{3}}\left(\frac{\sigma_t\sigma_c}{\sigma_t + \sigma_c}\right)^{1/2} \tag{65}$$

Special case: Cohesive-frictional material

For materials that can be characterized by cohesion c and friction angle ϕ, the Drucker-Prager parameters relate to these properties as:

$$\alpha = \frac{\tan\phi}{\sqrt{9 + 12\tan^2\phi}} \tag{66}$$

$$k = \frac{3c}{\sqrt{9 + 12\tan^2\phi}} \tag{67}$$

Comparison with von Mises:

The Drucker-Prager criterion is more general than von Mises:

- **Von Mises**: Pressure-independent, σ_y constant regardless of hydrostatic stress
- **Drucker-Prager**: Pressure-dependent, yield stress increases with confining pressure (positive α)

This makes Drucker-Prager more suitable for: - Soils and rocks (where cohesion and friction are important) - Concrete and other brittle materials - Powder-like materials - Any material where compressive strength exceeds tensile strength

Yield Surface Geometry

In principal stress space:

- **Von Mises**: Cylindrical surface centered on the hydrostatic axis
- **Drucker-Prager**: Conical surface (cone aligned with hydrostatic axis)

The Drucker-Prager cone opens wider with increasing pressure, indicating that higher hydrostatic pressure increases the allowable shear stress before yielding.

Practical Applications

Engineering design: The yield criterion determines the maximum allowable stress before plastic deformation initiates. Safety factors are typically applied:

$$\sigma_e \leq \frac{\sigma_y}{n_s} \tag{68}$$

where n_s is the safety factor (typically 1.5-3.0 depending on application).

Material selection: Materials with high yield stress are preferred for structural applications to limit plastic deformation and ensure elastic behavior under service loads.

> **Chapter Summary**
>
> This chapter covered yield criteria for predicting plastic failure:
>
> - **Yield criterion**: Function $F(\underline{\sigma})$ defining the boundary between elastic and plastic behavior
> - **Stress invariants**: $I_1 = \operatorname{tr} \underline{\sigma}$ (related to hydrostatic stress), J_2 (related to deviatoric stress)
> - **Von Mises criterion**: $\sigma_e = \sqrt{3J_2} = \sigma_y$; pressure-independent, widely used for metals
> - **Drucker-Prager criterion**: $\sqrt{J_2} + \alpha I_1 = k$; pressure-dependent, for soils and concrete
> - **Pure shear yield**: $\tau_y = \sigma_y/\sqrt{3}$ from von Mises criterion
> - **Safety factor**: Design requires $\sigma_e \leq \sigma_y/n_s$ with appropriate safety factor n_s
> - **Yield surface**: Cylindrical (von Mises) or conical (Drucker-Prager) in principal stress space
>
> Proper selection of yield criteria prevents unexpected plastic deformation in engineering structures.