Analysis of hyperstatic beams

Learning Objectives

- Distinguish between isostatic and hyperstatic (statically indeterminate) structures.
- Apply Castigliano's Second Theorem to compute deflections in beams.
- Use the Flexibility (Force) Method to solve first-degree statically indeterminate beams.
- Formulate and solve compatibility equations to find redundants and reactions.
- Compute shear and bending moment diagrams and visualize them with Python.

From isostatic to hyperstatic

In previous chapters, we dealt with **isostatic** (statically determinate) structures, where the equations of equilibrium ($\sum F = 0$, $\sum M = 0$) were sufficient to calculate all reactions and internal forces.

However, many microstructures are **hyperstatic** (statically indeterminate). This means they possess more supports or members than are strictly necessary for stability. To solve these systems, we must look beyond equilibrium and consider the **deformation** of the structure.

Hyperstatic beams can be solved by integrating four times the Euler Bernoulli equation, by applying the boundary conditions on displacements of the constraints. The anlytical method works well for analysing a single beam, but it becomes complicated for structures composed of several beams with internal constraints. This chapter introduces the Flexibility method as an alternative to solve hyperstatic structures.

Castigliano's Second Theorem

Alberto Castigliano (1847–1884) developed a powerful energy method for calculating displacements. His Second Theorem states that for a linear elastic structure, the partial derivative of the total strain energy, U, with respect to an applied force, P_i, is equal to the displacement, δ_i, at the point of application of that force in the direction of the force.

$$\delta_i = \frac{\partial U}{\partial P_i}$$

Application to Beams

For beams, the dominant strain energy arises from bending. The strain energy U is given by:

$$U = \int_0^L \frac{M(x)^2}{2EI}\, dx$$

Where: * $M(x)$ is the bending moment function. * E is the Young's Modulus. * I is the Moment of Inertia.

Applying the theorem, the deflection Δ at a point where a load P is applied is:

$$\Delta = \frac{\partial}{\partial P} \int_0^L \frac{M(x)^2}{2EI}\, dx$$

To simplify the integration, we can differentiate under the integral sign (Leibniz Integral Rule). This is often computationally more efficient:

$$\Delta = \int_0^L \frac{M(x)}{EI} \cdot \frac{\partial M(x)}{\partial P}\, dx$$

> **Note**
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> If you need to find the deflection at a point where there is no load, you must apply a "dummy load" Q, perform the partial derivative with respect to Q, and then set $Q = 0$ at the end.

The Flexibility Method (Force Method)

The Flexibility Method (also known as the Method of Consistent Deformations) relies on the principle of superposition. It treats the redundant reaction forces as unknown external loads.

The General Procedure

1. **Determine the Degree of Indeterminacy (n):**

$$n = R - E$$

   Where R is the number of reaction components and E is the number of available equilibrium equations.

2. **Define the Primary Structure:** Remove n redundant constraints (supports) to make the structure statically determinate (isostatic) and stable.

3. **Identify the Redundants:** The forces associated with the removed constraints are the "Redundants" ($X_1, X_2, ...$).

4. **Formulate Compatibility Equations:** The geometric continuity of the original structure must be maintained. For example, if we removed a support at Point B, the deflection at B in the real structure is zero.

   For a single degree of indeterminacy, the canonical equation is:

$$\Delta_{10} + f_{11} X_1 = 0$$

   - Δ_{10}: Displacement at point 1 caused by the *external loads* on the primary structure.
   - f_{11}: Displacement at point 1 caused by a *unit value* of the redundant force X_1 (Flexibility coefficient).
   - X_1: The magnitude of the redundant force.

5. **Solve for Redundants and Determine Reactions.**

Example: 2-Span Continuous Beam

Let us analyze a hyperstatic beam using the Flexibility Method.

Problem Statement: Consider a continuous beam of total length L. * **Supports:** - Left end (A, $x = 0$): Hinge (Pin). - Middle (B, $x = L/2$): Roller. - Right end (C, $x = L$): Roller. * **Loading:** A constant distributed line load q_0 pointing downward across the entire length. * **Stiffness:** Constant EI.

- Constraints: 1 hinghe + 2 rollers. Total = 4.
- Bodies: 1.
- **Degree:** $3 - 4 = -1$. The beam is statically indeterminate to the first degree.

We select the vertical reaction at the middle support (B) as our redundant force: - **Remove Support B.** - The system becomes a **Simply Supported Beam** spanning from A to C (Length L). - The redundant is the vertical force R_B (let's call it X_1).

We calculate the deflection at B (center) of the primary structure due to the external load q_0 only.

From standard beam tables (or integrating Castigliano's), the mid-span deflection of a simply supported beam under uniform load is:

$$\Delta_{B,0} = -\frac{5q_0 L^4}{384EI}$$

(The negative sign indicates downward deflection).

Now, we remove the load q_0 and apply a **unit force** $X_1 = 1$ (upward) at point B on the primary structure.

From standard beam tables, the mid-span deflection of a simply supported beam under a central point load $P = 1$ is:

$$f_{B,B} = \frac{1 \cdot L^3}{48EI}$$

In the real structure, the support at B is a roller, meaning the vertical displacement must be zero.

$$\Delta_{Total} = \Delta_{B,0} + f_{B,B} \cdot X_1 = 0$$

Substitute the values:

$$-\frac{5q_0 L^4}{384EI} + \left(\frac{L^3}{48EI}\right) X_1 = 0$$

Now we solve for the redundant force:

$$\frac{L^3}{48EI} X_1 = \frac{5q_0 L^4}{384EI}$$

Multiply both sides by $48EI/L^3$:

$$X_1 = \frac{5q_0 L \cdot 48}{384}$$

$$X_1 = \frac{5q_0 L}{8}$$

Therefore, the reaction at the middle support is $R_B = \frac{5}{8}q_0 L$.

Now that R_B is known, we use static equilibrium ($\sum F_y = 0$ and Symmetry) to find R_A and R_C.

1. **Symmetry:** Because the geometry and loading are symmetric, $R_A = R_C$.
2. **Sum of Forces:**

$$R_A + R_B + R_C = q_0 L$$

$$2R_A + \frac{5}{8}q_0 L = q_0 L$$

$$2R_A = q_0 L - \frac{5}{8}q_0 L = \frac{3}{8}q_0 L$$

$$R_A = \frac{3}{16}q_0 L$$

Final Results:

Support	Reaction Force
A (Left)	$\frac{3}{16}q_0 L$
B (Middle)	$\frac{5}{8}q_0 L$
C (Right)	$\frac{3}{16}q_0 L$

Internal Forces: Shear and Moment Diagrams

Now that the reactions are known, we determine the internal forces.

Analytical Functions

We define x from the left support A.

Interval 1: $0 \leq x < L/2$ **(Before the middle support)** * **Shear** $Q(x)$**:**

$$Q(x) = R_A - q_0 x = \frac{3}{16}q_0 L - q_0 x$$

* **Moment** $M(x)$**:**

$$M(x) = R_A x - \frac{q_0 x^2}{2} = \frac{3}{16}q_0 Lx - \frac{q_0 x^2}{2}$$

Interval 2: $L/2 < x \leq L$ (After the middle support) * **Shear $Q(x)$:**

$$Q(x) = R_A - q_0 x + R_B$$

* **Moment $M(x)$:**

$$M(x) = R_A x - \frac{q_0 x^2}{2} + R_B \left(x - \frac{L}{2} \right)$$